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ON CAPITAL ALLOCATION BY MINIMIZING MULTIVARIATE RISK INDICATORS

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Abstract. The issue of capital allocation in a multivariate context arises from the presence of dependence between the various risky activities which may generate a diversification effect. Several allocation methods in the literature are based on a choice of a univariate risk measure and an allocation principle, others on optimizing a multivariate ruin probability or some multivariate risk indicators. In this paper, we focus on the latter technique. Using an axiomatic approach, we study its coherence properties. We give some explicit results in mono periodic cases. Finally we analyze the impact of the dependence structure on the optimal allocation.

Introduction

The calculation of the regulatory economic capital, which is called the Solvency Capital Requirement in insurance, is well controlled and its methodology is almost imposed by the supervisory authorities of the sector. Nevertheless, the allocation of this capital may be considered as an internal exercise for each company, and constitutes a management choice whose success is a key factor for firm performance optimization. It can also be seen as an indicator of its good governance, especially for the multi branches firms.

The literature on the subject of the capital allocation methods is very rich. Several principles have been proposed over the last twenty years, the most important and most studied are the Shapley method, the Aumann-Shapley method and the Euler's method.
The Shapley method is based on cooperative game theory. It is described in detail in Denault's paper (2001) [11], where it is proved that this method, originally used to allocate the total cost between players in coalitional games context, can be easily adapted to solve problem of the overall risk allocation between segments.
Tasche devoted two papers [24] and [25], to the description of the Euler method, which is also found in the literature under the name of the gradient method. Euler method is based on the idea of allocating capital according to the infinitesimal marginal impact of each risk. This impact corresponds to the increase obtained on the overall risk, yielding an infinitely small increment in the risk i. The Euler method is very present in the literature, several papers analyze its properties (RORAC compatibility [7], [27], Coherence [6],...) under different assumptions (Tasche (2004) [23], Balog (2011) [3]). Its fame is due to the existence of economic arguments that can justify its use to develop allocation rules.
Finally, Aumann-Shapley method is a continuous generalization of Shapley method, its principle is based on the value introduced by Aumann and Shaplay in game theory. Denault [11] analyzes this method and its application to capital allocation.

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These three capital allocation principles rely on different risk measures, and the coherence of the allocation method depends on the properties of the selected risk measure. Several papers deal with capital allocation coherence based on the properties of the risk measure used, we quote as examples, Fischer (2003) [13], Bush and Dorfleitner (2008) [6], and Kalkbrener (2009) [17].

Other techniques have been proposed more recently for building optimal allocation methods, by minimizing some multivariate ruin probabilities, especially those defined by Cai and Li (2007) [8], or by minimizing some new multivariate risk indicators. In this context, Cénac et al [9], [10] defined three types of indicators, which take into account both the ruin severity at the branch level, and the impact of the dependence structure on this local severity. In the one-period case, these indicators can be considered as special cases of a general indicator family introduced in Dhaene et al (2012) [12].

Allocation by minimization of some real multivariate risk indicators can be used in a more general framework for modeling systemic risk, and also to determine the priorities of optimal stop-loss treaties that a reinsurer can offer according to the risk portfolios of its insurer customers, knowing that he covers his overall portfolio with a Stop-Loss treaty of priority u. This allocation technique can also help to measure the performance of calculating the groups capital requirement in the Swiss Solvency Test (SST), which provides a consistent framework both for legal branches and group solvency capital requirement.

The article is organized as follows. The first section is devoted to some definitions and notations that will be used throughout the rest of the article. Next, we study the coherence properties of the allocation by minimization of some specific indicators, and for a general choice of the penalties functions in Section 2. Section 3 is a presentation of some explicit formulas obtained for special model cases, the asymptotic behavior of the optimal allocation is also discussed in this section. In Section 4, we treat the question of the impact of the dependence structure on the composition of the optimal allocation, through the analysis of comonotonic cases and using some models of bivariate dependence with copulas.

1. Optimal allocation

In a multivariate risk framework, we consider a vectorial risk process $X^p = (X_1^p, \dots, X_d^p)$, where X_k^p corresponds to the losses of the k^{th} business line during the p^{th} period. We denote by R_p^k the reserve of the k^{th} line at time p, so: $R_p^k = u_k - \sum_{l=1}^{p} X_k^l$, where $u_k \in \mathbb{R}^+$ is the initial capital of the k^{th} business line, then $u = u_1 + \dots + u_d$ is the initial capital of the group, and d is the number of business lines.

Cénac et al (2012) [10] defined the two following multivariate risk indicators, given penalty functions g_k:

- the indicator I:

$$I(u_1, \dots, u_d) = \sum_{k=1}^{d} \mathbb{E}\left(\sum_{p=1}^{n} g_k(R_p^k) \mathbb{1}_{\{R_p^k<0\}} \mathbb{1}_{\{\sum_{j=1}^{d} R_p^j>0\}}\right),$$

- the indicator J:

$$J(u_1, \dots, u_d) = \sum_{k=1}^{d} \mathbb{E}\left(\sum_{p=1}^{n} g_k(R_p^k) \mathbb{1}_{\{R_p^k<0\}} \mathbb{1}_{\{\sum_{j=1}^{d} R_p^j<0\}}\right),$$

$g_k : \mathbb{R}^- \to \mathbb{R}^+$ are C^1, convex functions with $g_k(0) = 0,\ g_k(x) \geq 0$ for $x < 0,\ k = 1, \ldots, d$. They represent the cost that each branch has to pay when it becomes insolvent while the group is solvent for the I indicator, or while the group is also insolvent in the case of the J indicator.
They proposed to allocate some capital u by minimizing these indicators. The idea is to find an allocation vector $(u_1, \ldots, u_d)$ that minimizes the indicator such as $u = u_1 + \cdots + u_d$, where u is the initial capital that need to be shared among all branches.

The indicator I represents the expected sum of penalty amounts of local ruins, knowing that the group remains solvent. In the case of the indicator J, the local ruin severities are taken into account only in the case of group insolvency.

By using optimization stochastic algorithms, we may estimate the minimum of these risk indicators. Cénac et al (2012) [10] propose a Kiefer-Wolfowitz version of mirror algorithm as a convergent algorithm under general assumptions to find optimal allocation minimizing the indictor I. This algorithm is effective to solve the optimal allocation problem, especially, for a large number of business lines, and for allocation over several periods.

1.1. **Definitions and notations.** In this paper, we focus on the case of allocations on a single period ($n = 1$), since new regulation rules, such as Solvency 2, requires a justified allocation over a period of one year, and also to make a first computational approach, knowing that an annual allocation will be a more efficient choice for an insurer, that will allow it to integrate the changes that occurred at its risk portfolio and its dependence structure during a year of operation.

The following notations are used:

- ☐ The risk X_k corresponds to the losses of the k^{th} branch during one period. It is a positive random variable in our context.
- ☐ u is the initial capital of the firm.
- ☐ $\mathcal{U}_u^d = \{v = (v_1, \ldots, v_d) \in [0, u]^d, \sum_{i=1}^d v_i = u\}$ is the set of possible allocations. We define an allocation as an application $\mathbb{R}^+ \to (\mathbb{R}^+)^d$.
- ☐ For all $i \in \{1, \ldots, d\}$ let $\alpha_i = \frac{u_i}{u}$, then, $\sum_{i=1}^d \alpha_i = 1$.
- ☐ $\mathbb{1}_u^d = \{\alpha = (\alpha_1, \ldots, \alpha_d) \in [0, 1]^d, \sum_{i=1}^d \alpha_i = 1\}$ is the set of possible allocation percentages $\alpha_i = u_i/u$.
- ☐ For $(u_1, \ldots, u_d) \in \mathcal{U}_u^d$, we define the reserve of the k^{th} business line at the end of the period is: $R^k = u_k - X_k$, where u_k represents the part of capital allocated to the k^{th} branch.
- ☐ The aggregate sum of risks is: $S = \sum_{i=1}^d X_i$, and let $S^{-i} = \sum_{j=1; j\neq i}^d X_j$ for all $i \in \{1, \ldots, d\}$.
- ☐ F_Z is the distribution function of a random variable Z, $\bar{F}_Z$ is its survival function and f_Z its density function.

Definition 1.1 (Optimal allocation:)**.** Let X be a positive random vector of $\mathbb{R}^d$, $u \in \mathbb{R}^+$ and $\mathcal{K}_X : \mathcal{U}_u^d \to \mathbb{R}^+$ a multivariate risk indicator associated to X and u. An optimal allocation is defined by:

$$(u_1, \ldots, u_d) \in \underset{(v_1,\ldots,v_d) \in \mathcal{U}_u^d}{\arg\inf} \left\{ \mathcal{K}_X(v_1, \ldots, v_d) \right\}.$$

For risk indicators of the form $\mathcal{K}_X(v) = \mathbb{E}[S(X, v)]$, for a scoring function $S : \mathbb{R}^{+d} \times \mathbb{R}^{+d} \to \mathbb{R}^+$, this definition can be seen as an extension in a multivariate framework of the elicitability concept. Elicitability has been introduced by Gneiting (2011)[15], and studied recently for univariate risk measures, by Bellini and Bignozzi (2013)[4], Ziegel (2014)[28] and Steinwart et al (2014)[22], for examples.

We denote by $A_{X_1,\dots,X_d}(u) = (u_1, \dots, u_d)$, the optimal allocation of the amount u on the d risky branches in $\mathcal{U}_u^d$.

Assumptions: Throughout this paper, we will use the following assumptions:

- **H1:** The risk indicator admits an unique minimum in $\mathcal{U}_u^d$.
- **H2:** The functions g_k are differentiable and such that for all $k \in \{1, \dots, d\}$, $g'_k(u_k - X_k)$ admits a moment of order one, and (X_k, S) has a joint density distribution denoted by $f_{(X_k,S)}$.
- **H3:** The d risks have the same penalty function $g_k = g, \forall k \in \{1, \dots, d\}$.

The first assumption is verified when the indicator is strictly convex, this is particularly true when for at least one $k \in \{1, \dots, d\}$, g_k is strictly convex; and the joint density $f_{(X_k,S)}$ support contains $[0, u]^2$ (see [10]).

1.2. **Coherence properties.** In his article [11], Denault introduced the notion of a coherent allocation, fixing four axioms that must be verified by a principle of capital allocation, driven by coherent univariate risk measures, according to the criteria defined by Artzner et al (1999) [1], in order to be qualified as coherent. Our optimal capital allocation is not directly derived from a univariate risk measure, even if it is obtained by minimizing a multivariate risk indicator, we reformulate coherence axioms in our context.

1.2.1. *Coherence.* We follow Denault's idea to define a coherent capital allocation.

Definition 1.2 (Coherence)**.** A capital allocation $(u_1, \dots, u_d) \in \mathbb{R}^{+d}$ of an initial capital $u \in \mathbb{R}^+$ is coherent if it satisfies the following properties:

1. **Full allocation:** All of the capital $u \in \mathbb{R}^+$ must be allocated between the branches:
$$\sum_{i=1}^{d} u_i = u.$$
2. **Symmetry:** If the joint distribution of the vector $(X_1, \dots, X_d)$ is unchanged by permutation of the risks X_i and X_j, then the allocation remains also unchanged by this permutation, and the i^{th} and j^{th} business lines both make the same contribution to the risk capital:
$$(X_1, \dots, X_{i-1}, X_i, X_{i+1} \dots, X_{j-1}, X_j, X_{j+1}, \dots, X_d) \stackrel{\mathcal{L}}{=} (X_1, \dots, X_{i-1}, X_j, X_{i+1} \dots, X_{j-1}, X_i, X_{j+1}, \dots, X_d),$$
then $u_i = u_j$.
3. **Riskless allocation:** For a deterministic risk $X = c$, where the constant $c \in \mathbb{R}^+$:
$$A_{X,X_1,\dots,X_d}(u) = (c, A_{X_1,\dots,X_d}(u - c)).$$
This property means that the optimal allocation method relates only risky branches, the presence of a deterministic risk has no impact on the share allocated to the risky branches.
4. **Sub-additivity:** $\forall M \subseteq \{1, \dots, d\}$, let $(u^*, u_1^*, \dots, u_r^*) = A_{\sum_{i\in M} X_i, X_{j\in\{1,\dots,d\}\setminus M}}(u)$, where $r = d - card(M)$ and $(u_1, \dots, u_d) = A_{X_1,\dots,X_d}(u)$:
$$u^* \leq \sum_{i\in M} u_i.$$
This property mains that the optimal allocation takes into account the diversification gain. It is related to the *no undercut* property defined by Denault, which has no sense in our context.

5. Comonotonic additivity: For $r \leqslant d$ comonotonic risks,

$$A_{X_{i_{i\in\{1,\dots,d\}\setminus CR}}, \sum_{k\in CR} X_k}(u) = (u_{i_{i\in\{1,\dots,d\}\setminus CR}}, \sum_{k\in CR} u_k),$$

where $(u_1,\dots,u_d) = A_{X_1,\dots,X_d}(u)$ is the optimal allocation of u on the d risks $(X_1,\dots,X_d)$ and CR denote the set of the r comonotonic risk indexes.
The concept of comonotonic random variables is related to the studies of Hoeffding (1940) [16] and Fréchet (1951) [14]. Here we use the definition of comonotonic risks as it was first mentioned in the actuarial literature in Borch (1962) [5].

1.2.2. *Other desirable properties.* We define also some desirable properties that an optimal allocation must naturally satisfy. These properties are based on the ideas presented by Artzner et al (1999) [1] for coherent risk measures and on the axiomatic characterization of coherent capital allocations given by Kalkbrener (2009) [17].

Definition 1.3 (Positive homogeneity)**.** An optimal allocation is positively homogeneous, if for any $\alpha \in \mathbb{R}^+$, it satisfies:

$$A_{\alpha X_1,\dots,\alpha X_d}(\alpha u) = \alpha A_{X_1,\dots,X_d}(u).$$

In other words, a capital allocation method is positively homogeneous, if it is insensitive to the cash changes.

Definition 1.4 (Translation invariance)**.** An optimal allocation is invariant by translation, if for all $(a_1,\dots,a_d) \in \mathbb{R}^d$, it satisfies:

$$A_{X_1-a_1,\dots,X_d-a_d}(u) = A_{X_1,\dots,X_d}\left(u + \sum_{k=1}^{d} a_k\right) - (a_1,\dots,a_d).$$

The translation invariance property shows that the impact of an increase (decrease) of a risk by a constant amount of its share of allocation of the capital u, boils down to an increase (decrease) of its share in the allocation of such capital decreased (increased) by the same amount.

Definition 1.5 (Continuity)**.** An optimal allocation is continuous, if for all $i \in \{1,\dots,d\}$:

$$\lim_{\epsilon\to 0} A_{X_1,\dots,(1+\epsilon)X_i,\dots,X_d}(u) = A_{X_1,\dots,X_i,\dots,X_d}(u).$$

This property reflects the fact that a small change to the risk of a business line, have only limited effect on the capital part that we attribute to it.

Let's recall the definition of the order stochastic dominance, as it is presented in Shaked and Shanthikumar (2007)[21]. For random variables X and Y, Y first-order stochastically dominates X if and only if:

$$\bar{F}_X(x) \leq \bar{F}_Y(x), \quad \forall x \in \mathbb{R}^+,$$

and in this case we denote: $X \leq_{st} Y$.
This definition is also equivalent to the following one:

$$X \leq_{st} Y \Leftrightarrow \mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)], \text{for all } u \text{ increasing function}$$

Definition 1.6 (Monotonicity)**.** An optimal allocation satisfies the monotonicity property, if for $(i,j) \in \{1,\dots,d\}^2$:

$$X_i \leq_{st} X_j \Rightarrow u_i \leq u_j.$$

The monotonicity is a natural requirement, it reflects the fact that if a branch X_j is riskier than branch X_i. Then, it is natural to allocate more capital to the risk X_j.

The RORAC compatibility property defined by Dirk Tasche [24] loses its meaning in absence of the risk measure used in the construction of the allocation method.

1.3. **Optimality conditions.** In this section, we focus on the optimality condition for the indicators I and J.
For an initial capital u, and an optimal allocation minimizing the multivariate risk indicator I, we seek $u^* \in \mathbb{R}^d_+$ such that:

$$I(u^*) = \inf_{v_1+\cdots+v_d=u} I(v), \quad v \in \mathbb{R}^d_+.$$

Under assumption H2, the risk indicators I and J are differentiable, and in this case, we can calculate the following gradients:

$$(\nabla I(v))_i = \sum_{k=1}^{d} \int_{v_k}^{+\infty} g_k(v_k - x) f_{X_k,S}(x,u)dx + \mathbb{E}[g_i'(v_i - X_i)\mathbb{1}_{\{X_i > v_i\}}\mathbb{1}_{\{S \leq u\}}]$$

$$\text{and,} \quad (\nabla J(v))_i = \sum_{k=1}^{d} \int_{v_k}^{+\infty} g_k(v_k - x) f_{X_k,S}(x,u)dx + \mathbb{E}[g_i'(v_i - X_i)\mathbb{1}_{\{X_i > v_i\}}\mathbb{1}_{\{S \geq u\}}].$$

Under H1 and H2, using the Lagrange multipliers method, we obtain an optimality condition verified by the unique solution to this optimization problem:

$$\mathbb{E}[g_i'(u_i - X_i)\mathbb{1}_{\{X_i > u_i\}}\mathbb{1}_{\{S \leq u\}}] = \mathbb{E}[g_i'(u_j - X_j)\mathbb{1}_{\{X_j > u_j\}}\mathbb{1}_{\{S \leq u\}}], \quad \forall j \in \{1,\ldots,d\}^2. \tag{1.1}$$

A natural choice for penalty functions is the ruin severity: $g_k(x) = |x|$. In that case, and if the joint density $f_{(X_k,S)}$ support contains $[0,u]^2$, for at least one $k \in \{1,\ldots,d\}$, our optimization problem has a unique solution.
We may write the indicators as follows:

$$\begin{aligned} I(u_1,\ldots,u_d) &= \sum_{k=1}^{d} \mathbb{E}\left(|R^k|\mathbb{1}_{\{R^k<0\}}\mathbb{1}_{\{\sum_{i=1}^d R^i \geq 0\}}\right) \\ &= \sum_{k=1}^{d} \mathbb{E}\left((X_k - u_k)\mathbb{1}_{\{X_k > u_k\}}\mathbb{1}_{\{\sum_{i=1}^d X_i \leq u\}}\right) = \sum_{k=1}^{d} \mathbb{E}\left((X_k - u_k)^+\mathbb{1}_{\{S \leq u\}}\right), \end{aligned}$$

and,

$$\begin{aligned} J(u_1,\ldots,u_d) &= \sum_{k=1}^{d} \mathbb{E}\left(|R^k|\mathbb{1}_{\{R^k<0\}}\mathbb{1}_{\{\sum_{i=1}^d R^i \leq 0\}}\right) \\ &= \sum_{k=1}^{d} \mathbb{E}\left((X_k - u_k)\mathbb{1}_{\{X_k > u_k\}}\mathbb{1}_{\{\sum_{i=1}^d X_i \geq u\}}\right) = \sum_{k=1}^{d} \mathbb{E}\left((X_k - u_k)^+\mathbb{1}_{\{S \geq u\}}\right). \end{aligned}$$

The respective components of the gradient of these indicators are of the form:

$$K_I - \mathbb{P}\left(X_1 > u_1, \sum_{j=1}^{d} X_j \leq u\right), \ldots, K_I - \mathbb{P}\left(X_d > u_d, \sum_{j=1}^{d} X_j \leq u\right),$$

and,

$$K_J - \mathbb{P}\left(X_1 > u_1, \sum_{j=1}^{d} X_j \geq u\right), \ldots, K_J - \mathbb{P}\left(X_d > u_d, \sum_{j=1}^{d} X_j \geq u\right),$$

where,

$$K_I = K_J = \sum_{k=1}^{d} \int_{u_k}^{+\infty} (x - u_k) f_{X_k,S}(x,u)dx.$$

Using the Lagrange multipliers to solve our convex optimization problem under the only constraint $u_1 + u_2 + \cdots + u_d = u$, the following optimality conditions are obtained from 1.1 in the special case where $g_k(x) = |x|$:

$$\mathbb{P}\left(X_i > u_i, S \leq u\right) = \mathbb{P}\left(X_j > u_j, S \leq u\right), \forall (i,j) \in \{1, 2, \ldots, d\}^2. \tag{1.2}$$

For the J indicator, this condition can be written:

$$\mathbb{P}\left(X_i > u_i, S \geq u\right) = \mathbb{P}\left(X_j > u_j, S \geq u\right), \forall (i,j) \in \{1, 2, \ldots, d\}^2. \tag{1.3}$$

Some explicit and semi-explicit formulas for the optimal allocation can be obtained with this optimality condition. Our problem reduces to the study of this allocation depending on the nature of the distributions of the risk X_k and on the form of dependence between them.

1.4. **Main theoretical results.** In section 2, we shall prove the following theorem.

Theorem 1.7. *In the case of penalty functions $g_k(x) = |x|$ $\forall k \in \{1, \ldots, d\}$, and for continuous random vector $(X_1, \ldots, X_d)$, such that the joint density $f_{(X_k,S)}$ support contains $[0,u]^2$, for at least one $k \in \{1, \ldots, d\}$, the optimal allocation by minimization of the indicators I and J is a symmetric riskless full allocation. It satisfies the properties of comonotonic additivity, positive homogeneity, translation invariance, monotonicity, and continuity.*

The main theoretical result of section 3 is Theorem 3.7, from which the asymptotic (as $u \to +\infty$) behavior of the optimal allocation may be derived for large classes of independent distributions.

2. Properties of the optimal allocation method

In what follows, we show that the capital allocation minimizing the indicator I, satisfies the coherence axioms of Definition 1.2, except the sub-additivity. We show also that it satisfies other desirable properties in the second subsection. The same holds for the indicator J.

2.1. **Coherence.** Firstly, the *full allocation* axiom is verified by construction, since any optimal allocation satisfies the equality:

$$\sum_{i=1}^{d} u_i = u.$$

Proposition 2.1 shows that the optimal allocation satisfies the symmetry property.

Proposition 2.1 (**Symmetry**). *Under H1, if for $(i,j) \in \{1, 2, \ldots, d\}^2$, $i \neq j$, the couples (X_i, S^{-i}) and (X_j, S^{-j}) are identically distributed and the penalty functions g_i and g_j are the same $g_i = g_j$, then:*

$$u_i = u_j.$$

Proof. Let $(i \neq j) \in \{1, 2, \ldots, d\}^2$ be such that (X_i, S^{-i}) and (X_j, S^{-j}) have the same distribution and the same penalty function $g_i = g_j = g$. If $u_i \neq u_j$, we may assume $i < j$, and denote:

$$(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_d) = A_{X_1,\ldots,X_i,\ldots,X_j,\ldots,X_d}(u),$$

then,

$$I(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_d) = \inf_{v \in \mathcal{U}_u^d} I(v) = \inf_{v \in \mathcal{U}_u^d} \sum_{k=1}^{d} \mathbb{E}\left(g_k(v_k - X_k) \mathbb{1}_{\{X_k > v_k\}} \mathbb{1}_{\{S \leq u\}}\right).$$

On the other hand, and since $g_i = g_j = g$ and $(X_i, S^{-i}) \sim (X_j, S^{-j})$, then:

$$\begin{aligned}
I(u_1,\dots,u_{i-1},u_j,u_{i+1},\dots,u_{j-1},u_i,u_{j+1},\dots,u_d) &= \sum_{k=1,k\neq i,k\neq j}^{d} \mathbb{E}\left(g_k(u_k - X_k)\mathbb{1}_{\{X_k>u_k\}}\mathbb{1}_{\{S\le u\}}\right)\\
&+ \mathbb{E}\left(g(u_i - X_i)\mathbb{1}_{\{X_i>u_i\}}\mathbb{1}_{\{S\le u\}}\right)\\
&+ \mathbb{E}\left(g(u_j - X_j)\mathbb{1}_{\{X_j>u_j\}}\mathbb{1}_{\{S\le u\}}\right)\\
&= I(u_1,\dots,u_i,\dots,u_j,\dots,u_d).
\end{aligned}$$

From H1, the indicator I admits an unique minimum in $\mathcal{U}_u^d$, we deduce that:

$$(u_1,\dots,u_i,\dots,u_j,\dots,u_d) = (u_1,\dots,u_{i-1},u_j,u_{i+1},\dots,u_{j-1},u_i,u_{j+1},\dots,u_d).$$

We conclude that $u_i = u_j$. □

Corollary 2.2. *Under Assumptions H1 and H3, if $(X_1,\dots,X_d)$ is an exchangeable random vector, then the allocation by minimizing I and J indicators is the same and is given by:*

$$A_{X_1,\dots,X_d}(u) = \left(\frac{u}{d},\frac{u}{d},\dots,\frac{u}{d}\right).$$

The following proposition shows that the optimal allocation verifies the Riskless allocation axiom.

Proposition 2.3 (**Riskless Allocation**). *Under Assumptions H1 and H3, and for 1-homogeneous penalty functions, for any $c \in \mathbb{R}$:*

$$A_{c,X_1,\dots,X_d}(u) = (c, A_{X_1,\dots,X_d}(u-c)),$$

where $(c, A_{X_1,\dots,X_d}(u-c))$ is the concatenated vector of c and the vector $A_{X_1,\dots,X_d}(u-c)$.

Proof. The presence of a discrete distribution makes the indicator I not differentiable, so we cannot use neither the gradient, nor the optimality condition obtained in the case of existence of joined densities.
Let, $(u^*, u_1^*, \cdots, u_d^*) = A_{c,X_1,\dots,X_d}(u)$ and $(u_1,\cdots,u_d) = A_{X_1,\dots,X_d}(u-c)$.
We denote $S = \sum_{i=1}^d X_i$, and the common penalty function $g = g_k, \forall k \in \{1,\dots,d\}$, the function g is convex on $\mathbb{R}^-$ and $g(0)=0$, we deduce that g is also positively homogeneous.
We distinguish between three possibilities:

- *Case 1: $u^* < c$*
  In this case,

$$\begin{aligned}
I(u^*, u_1^*,\dots,u_d^*) &= \inf_{v\in\mathcal{U}_u^{d+1}} I(v) = \inf_{v\in\mathcal{U}_u^{d+1}} \sum_{k=0}^{d} \mathbb{E}\left(g(v_k - X_k)\mathbb{1}_{\{X_k>v_k\}}\mathbb{1}_{\{S\le u-c\}}\right)\\
&= \mathbb{E}\left(g(u^*-c)\mathbb{1}_{\{S\le u-c\}}\right) + \sum_{k=1}^{d} \mathbb{E}\left(g(u_k^* - X_k)\mathbb{1}_{\{X_k>u_k^*\}}\mathbb{1}_{\{S\le u-c\}}\right),
\end{aligned}$$

  for all $k \in \{1,\dots,d\}$ we put, for example, $\alpha_k = \alpha = \frac{u^*-c}{d} < 0$, and since the function g is convex and $g(0)=0$, it satisfies for all real $0<\beta<1$, $g(\beta x) \le \beta g(x), \forall x \in \mathbb{R}^-$. Then:

$$\begin{aligned}
I(u^*, u_1^*,\dots,u_d^*) &\ge \mathbb{E}\left(d\cdot g\left(\frac{u^*-c}{d}\right)\mathbb{1}_{\{S\le u-c\}}\right) + \sum_{k=1}^{d} \mathbb{E}\left(g(u_k^* - X_k)\mathbb{1}_{\{X_k>u_k^*\}}\mathbb{1}_{\{S\le u-c\}}\right)\\
&= \mathbb{E}\left(d\cdot g\left(-(-\alpha)_+\right)\mathbb{1}_{\{S\le u-c\}}\right) + \sum_{k=1}^{d} \mathbb{E}\left(g(-(X_k - u_k^*)_+)\mathbb{1}_{\{S\le u-c\}}\right)\\
&= \sum_{k=1}^{d} \mathbb{E}\left([g(-(X_k-u_k^*)_+) + g(-(-\alpha_k)_+)]\mathbb{1}_{\{S\le u-c\}}\right),
\end{aligned}$$

$x \to g(-(x)_+)$ is also a 1-homogeneous convex function, then:

$$I(u^*, u_1^*, \dots, u_d^*) \geq \sum_{k=1}^{d} \mathbb{E}\left(g(-(X_k - (u_k^* + \alpha_k))_+)\mathbb{1}_{\{S \leq u-c\}}\right),$$

we remark that $\sum_{k=1}^{d}(u_k^* + \alpha_k) = u - c$, then $(u_1^* + \alpha, \dots, u_d^* + \alpha) \in \mathcal{U}_{u-c}^d$.
So,

$$\begin{aligned}
I(u^*, u_1^*, \dots, u_d^*) &\geq \sum_{k=1}^{d} \mathbb{E}\left(g((u_k^* + \alpha_k) - X_k))\mathbb{1}_{\{X_k > u_k^* + \alpha_k\}}\mathbb{1}_{\{S \leq u-c\}}\right) \\
&\geq \inf_{v \in \mathcal{U}_{u-c}^d} \sum_{k=1}^{d} \mathbb{E}\left(g(v_k - X_k)\mathbb{1}_{\{X_k > v_k\}}\mathbb{1}_{\{S \leq u-c\}}\right) \\
&= \sum_{k=1}^{d} \mathbb{E}\left(g(u_k - X_k)^+\mathbb{1}_{\{S \leq u-c\}}\right) \\
&= I(c, u_1, \dots, u_d),
\end{aligned}$$

then,

$$I(u^*, u_1^*, \dots, u_d^*) \geq I(c, u_1, \dots, u_d).$$

That is contradictory with the uniqueness of the minimum on the set $\mathcal{U}_u^{d+1}$.

- *Case 2:* $u^* > c$
  We have :

$$I(u^*, u_1^*, \dots, u_d^*) = \sum_{k=1}^{d} \mathbb{E}\left(g(u_k^* - X_k)\mathbb{1}_{\{X_k > u_k^*\}}\mathbb{1}_{\{S \leq u-c\}}\right),$$

and,

$$I(c, u_1, \dots, u_d) = \sum_{k=1}^{d} \mathbb{E}\left(g(u_k - X_k)\mathbb{1}_{\{X_k > u_k\}}\mathbb{1}_{\{S \leq u-c\}}\right).$$

Let $\alpha = \frac{u^* - c}{d} > 0$, we remark that,$(u_1^* + \alpha, \dots, u_d^* + \alpha) \in \mathcal{U}_{u-c}^d$, and that the penalty function g is decreasing on $\mathbb{R}^-$ because $g''(x) \geq 0, \forall x \in \mathbb{R}^-$ and $g'(0^+) = 0$. Then,

$$\begin{aligned}
I(c, u_1, \dots, u_d) &= \sum_{k=1}^{d} \mathbb{E}\left(g(-(X_k - u_k)_+)\mathbb{1}_{\{S \leq u-c\}}\right) \\
&= \inf_{v \in \mathcal{U}_{u-c}^d} \sum_{k=1}^{d} \mathbb{E}\left(g(v_k - X_k)\mathbb{1}_{\{X_k > v_k\}}\mathbb{1}_{\{S \leq u-c\}}\right) \\
&\leq \sum_{k=1}^{d} \mathbb{E}\left(g(-(X_k - (u_k^* + \alpha))_+)\mathbb{1}_{\{S \leq u-c\}}\right) \\
&< \sum_{k=1}^{d} \mathbb{E}\left(g(-(X_k - u_k^*))_+\mathbb{1}_{\{S \leq u-c\}}\right) \\
&= \sum_{k=1}^{d} \mathbb{E}\left(g(u_k^* - X_k)\mathbb{1}_{\{X_k > u_k^*\}}\mathbb{1}_{\{S \leq u-c\}}\right) = I(u^*, u_1^*, \dots, u_d^*).
\end{aligned}$$

That is contradictory with the fact that $I(u^*, u_1^*, \dots, u_d^*) = \inf_{v \in \mathcal{U}_u^{d+1}} I(v)$.
We deduce that the only possible case is the third one $u^* = c$.

- *Case 3:* $u^* = c$
  The uniqueness of the minimum implies that:

$$
\begin{aligned}
(u^*, u_1^*, \ldots, u_d^*) = (c, u_1^*, \ldots, u_d^*) &= \underset{\mathcal{U}_u^{d+1}}{\arg\min} \sum_{k=1}^{d} \mathbb{E}[g(u_k - X_k)\mathbb{1}_{\{X_k > u_k\}}\mathbb{1}_{\{S \le u-c\}}] \\
&= \underset{\mathcal{U}_{u-c}^{d}}{\arg\min} \sum_{k=1}^{d} \mathbb{E}[g(u_k - X_k)\mathbb{1}_{\{X_k > u_k\}}\mathbb{1}_{\{S \le u-c\}}] \\
&= (c, u_1, \ldots, u_d).
\end{aligned}
$$

  Finally, we have proven that:

$$(u^*, u_1^*, \ldots, u_d^*) = (c, u_1, \ldots, u_d).$$

□

Lemma 2.4 is related to the sub-additivity property. It will be used in the proof of the comonotonic additivity property.

Lemma 2.4. *Under Assumptions H1,H2 and H3, and For all $(i,j) \in \{1, \ldots, d\}^2$, and where, $x.e_i$ is the dot product of the vector $x \in \mathbb{R}^d$ and the i^{th} component of the canonical basis of $\mathbb{R}^d$.*

- *if* $A_{X_1,\ldots,X_{i-1},X_i+X_j,X_{i+1},\ldots,X_{j-1},X_{j+1},\ldots,X_d}(u).e_i < A_{X_1,\ldots,X_d}(u).(e_i + e_j)$, *then:*
  $\forall k \in \{1, \ldots, d\} \setminus i, j, \quad A_{X_1,\ldots,X_{i-1},X_i+X_j,X_{i+1},\ldots,X_{j-1},X_{j+1},\ldots,X_d}(u).e_k > A_{X_1,\ldots,X_d}(u).e_k,$
- *if* $A_{X_1,\ldots,X_{i-1},X_i+X_j,X_{i+1},\ldots,X_{j-1},X_{j+1},\ldots,X_d}(u).e_i > A_{X_1,\ldots,X_d}(u).(e_i + e_j)$, *then:*
  $\forall k \in \{1, \ldots, d\} \setminus i, j, \quad A_{X_1,\ldots,X_{i-1},X_i+X_j,X_{i+1},\ldots,X_{j-1},X_{j+1},\ldots,X_d}(u).e_k < A_{X_1,\ldots,X_d}(u).e_k,$
- *if* $A_{X_1,\ldots,X_{i-1},X_i+X_j,X_{i+1},\ldots,X_{j-1},X_{j+1},\ldots,X_d}(u).e_i = A_{X_1,\ldots,X_d}(u).(e_i + e_j)$, *then:*
  $\forall k \in \{1, \ldots, d\} \setminus i, j, \quad A_{X_1,\ldots,X_{i-1},X_i+X_j,X_{i+1},\ldots,X_{j-1},X_{j+1},\ldots,X_d}(u).e_k = A_{X_1,\ldots,X_d}(u).e_k.$

Proof. In order to simplify the notation, and without loss of generality, we assume $i = d-1$ and $j = d$. We put,$(u_1, \ldots, u_{d-1}, u_d) = A_{X_1,\ldots,X_d}(u)$ and $(u_1^*, \ldots, u_{d-2}^*, u_{d-1}^*) = A_{X_1,\ldots,X_{d-2},X_{d-1}+X_d}(u)$. The optimality condition for $(u_1, \ldots, u_{d-1}, u_d)$ is given $\forall (i,j) \in \{1, \ldots, d\}^2$ by equation 1.1:

$$\mathbb{E}[g_i'(u_i - X_i)\mathbb{1}_{\{X_i > u_i\}}\mathbb{1}_{\{S \le u\}}] = \mathbb{E}[g_i'(u_j - X_j)\mathbb{1}_{\{X_j > u_j\}}\mathbb{1}_{\{S \le u\}}] = \lambda,$$

and for $(u_1^*, \ldots, u_{d-2}^*, u_{d-1}^*)$ is $\forall i \in \{1, \ldots, d-2\}$

$$\mathbb{E}[g_i'(u_i^* - X_i)\mathbb{1}_{\{X_i > u_i^*\}}\mathbb{1}_{\{S \le u\}}] = \mathbb{E}[g_i'(u_{d-1}^* - (X_{d-1} + X_d))\mathbb{1}_{\{X_d + X_{d-1} > u_{d-1}^*\}}\mathbb{1}_{\{S \le u\}}] = \lambda^*.$$

Now, we suppose that $u_{d-1}^* > u_d + u_{d-1}$. In this case there exists $k \in \{1, \ldots, d-2\}$ such that $u_k^* < u_k$, and since the function $x \to g'(-(x)_+)$ is decreasing on $\mathbb{R}^+$, then:

$$\mathbb{E}[g_i'(u_k - X_k)\mathbb{1}_{\{X_k > u_k\}}\mathbb{1}_{\{S \le u\}}] = \lambda < \mathbb{E}[g_i'(u_k^* - X_k)\mathbb{1}_{\{X_k > u_k^*\}}\mathbb{1}_{\{S \le u\}}] = \lambda^*$$

we deduce from this that for all $k \in 1, \ldots, d-2 : \ u_k^* < u_k$.
The proof is the same if we suppose that $u_{d-1}^* < u_d + u_{d-1}$, and the additive case is a corollary of the two previous ones. □

Proposition 2.5 (Comonotonic additivity)**.** *Under Assumption H2, and for $g_k(x) = |x|$, for all $k \in \{1, \ldots, d\}$, if $r \le d$ risks $X_{i_{i \in CR}}$ are comonotonics, then:*

$$A_{X_{i_{i \in \{1,\ldots,d\} \setminus CR}}, \sum_{k \in CR} X_k}(u) = (u_{i_{i \in \{1,\ldots,d\} \setminus CR}}, \sum_{k \in CR} u_k),$$

where $(u_1, \ldots, u_d) = A_{X_1,\ldots,X_d}(u)$ is the optimal allocation of u on the d risks $(X_1, \ldots, X_d)$, $A_{X_{i_{i \in \{1,\ldots,d\} \setminus CR}}, \sum_{k \in CR} X_k}(u)$ is the optimal allocation of u on the $n-d+1$ risks $(X_{i_{i \in \{1,\ldots,d\} \setminus CR}}, \sum_{k \in CR} X_k)$, and CR denote the set of the r comonotonic risk indexes.

Proof. For $(i,j) \in \{1,\dots,d\}^2$, if X_i and X_j are comonotonic risks, then, there exists an increasing non negative function h such that $X_i = h(X_j)$, and we remark that h is strictly increasing under Assumption H2. Let f be the function $x \to f(x) = x + h(x)$, so that $X_i + X_j = f(X_j)$.
We denote $(u_1,\dots,u_d) = A_{X_1,\dots,X_d}(u)$ and $(u_1^*,\dots,u_{d-1}^*) = A_{X_{i\in\{1,\dots,d\}\setminus\{i,j\}},X_i+X_j}(u)$, then, $A_{X_{i\in\{1,\dots,d\}\setminus\{i,j\}},X_i+X_j}(u).e_{d-1} = u_{d-1}^*$ and $A_{X_1,\dots,X_d}(u).(e_i + e_j) = u_i + u_j$.
From the optimality condition for the allocation $A_{X_1,\dots,X_d}(u)$, given in Equation 1.2:

$$\mathbb{P}(X_i \geq u_i, S \leq u) = \mathbb{P}(X_j \geq u_j, S \leq u),$$

we deduce that $u_i = h(u_j)$ and that $u_i + u_j = f(u_j)$.
If there exists $k \in \{1,\dots,d\} \setminus \{i,j\}$, such that $u_k^* < u_k$, then $\forall k \in \{1,\dots,d\} \setminus \{i,j\}$:

$$\mathbb{P}(X_k \geq u_k^*, S \leq u) > \mathbb{P}(X_k \geq u_k, S \leq u),$$

so,

$$\begin{aligned}
\mathbb{P}(X_i + X_j \geq u_{d-1}^*, S \leq u) &= \mathbb{P}(X_j \geq f^{-1}(u_{d-1}^*), S \leq u) \\
&= \mathbb{P}(X_k \geq u_k^*, S \leq u) \\
&> \mathbb{P}(X_k \geq u_k, S \leq u) \\
&= \mathbb{P}(X_j \geq u_j, S \leq u),
\end{aligned}$$

finally, we deduce that: $f^{-1}(u_{d-1}^*) < u_j$, then $u_{d-1}^* < f(u_j) = u_i + u_j$ and, $\sum_{k\in\{1,\dots,d\}\setminus\{i,j\}} u_k^* < \sum_{k\in\{1,\dots,d\}\setminus\{i,j\}} u_k$ which is absurd.
In the same way, the case $u_k < u_k^*$ for $k \in \{1,\dots,d\} \setminus \{i,j\}$ leads to the contradiction.
Using Lemma 2.4, and under Assumption H3, we deduce the optimal allocation for the other risks X_k, $k \in \{1,\dots,d\} \setminus \{i,j\}$.
The additivity property for two comonotonic risks is trivially generalizable to several comonotonic risks. □

Concerning the sub-additivity property, we have not yet managed to build a demonstration for this property. However, the simulations using the optimization algorithm presented in Cénac et al (2012) [10], seem to confirm the sub-additivity of the allocation by minimizing the indicators I and J, even for classic examples of non sub-additivity of the risk measure VaR.

Remark 2.6. The previous properties have been demonstrated for the optimal allocation by minimizing the risk indicator I, they can be demonstrated with the same arguments for the optimal allocation by minimization of the indicator J.

2.2. **Other desirable properties.** In this section, we show that the optimal allocation by minimization of the indicators I and J satisfies some desirable properties. We consider the allocation by minimizing the multivariate risk indicator I, the demonstrations are almost the same in the case of the indicator J.

The following proposition shows that the optimal allocation by minimization indicators I and J satisfies the property of positive homogeneity.

Proposition 2.7 (Positive homogeneity)**.** *Under Assumption H1, and for 1-homogeneous penalty functions g_k, $k \in \{1,\dots,d\}$, for any $\alpha \in \mathbb{R}^+$:*

$$A_{\alpha X_1,\dots,\alpha X_d}(\alpha u) = \alpha A_{X_1,\dots,X_d}(u).$$

Proof. Since the penalty functions are convex and 1-homogeneous, then, for any $\alpha \in \mathbb{R}^{*+}$:

$$\begin{aligned}
A_{\alpha X_1,\dots,\alpha X_d}(\alpha u) &= \underset{(u_1^*,\dots,u_d^*)\in\mathcal{U}^d_{\alpha u}}{\arg\min} \sum_{k=1}^d \mathbb{E}[g_k(u_k^* - \alpha X_k)\mathbb{1}_{\{\alpha X_k > u_k^*\}}\mathbb{1}_{\{\alpha S \leq \alpha u\}}] \\
&= \underset{(u_1^*,\dots,u_d^*)\in\mathcal{U}^d_{\alpha u}}{\arg\min} \sum_{k=1}^d \alpha\mathbb{E}[g_k\left(\frac{u_k^*}{\alpha} - X_k\right)\mathbb{1}_{\{X_k > \frac{u_k^*}{\alpha}\}}\mathbb{1}_{\{S \leq u\}}] \\
&= \underset{(u_1^*,\dots,u_d^*)\in\mathcal{U}^d_{\alpha u}}{\arg\min} \sum_{k=1}^d \mathbb{E}[g_k\left(\frac{u_k^*}{\alpha} - X_k\right)\mathbb{1}_{\{X_k > \frac{u_k^*}{\alpha}\}}\mathbb{1}_{\{S \leq u\}}] \\
&= \alpha \underset{(u_1,\dots,u_d)\in\mathcal{U}^d_{u}}{\arg\min} \sum_{k=1}^d \mathbb{E}[g_k(u_k - X_k)\mathbb{1}_{\{X_k > u_k\}}\mathbb{1}_{\{S \leq u\}}] \\
&= \alpha A_{X_1,\dots,X_d}(u).
\end{aligned}$$

□

Proposition 2.8 (Translation invariance)**.** *Under Assumptions H1, H2 and for all* $(a_1,\dots,a_d) \in \mathbb{R}^d$*, such that the joint density* $f(X_k, S)$ *support contains* $[0, u + \sum_{k=1}^d a_k]^2$*, for all* $k \in \{1,\dots,d\}$*:*

$$A_{X_1-a_1,\dots,X_d-a_d}(u) = A_{X_1,\dots,X_d}\left(u + \sum_{k=1}^d a_k\right) - (a_1,\dots,a_d).$$

Proof. We denote by $(u_1^*,\dots,u_d^*)$ the optimal allocation $A_{X_1-a_1,\dots,X_d-a_d}(u)$, and by $(u_1,\dots,u_d)$ the optimal allocation $A_{X_1,\dots,X_d}\left(u + \sum_{k=1}^d a_k\right)$.
Using the optimality condition (1.1), $(u_1^*,\dots,u_d^*)$ is the unique solution in $\mathcal{U}^d_u$ of the following equation system:

$$\mathbb{E}[g_i'(u_i^* - (X_i - a_i))\mathbb{1}_{\{X_i - a_i > u_i^*\}}\mathbb{1}_{\{S-a\leq u\}}] = \mathbb{E}[g_j'(u_j^* - (X_j - a_j))\mathbb{1}_{\{X_j - a_j > u_j^*\}}\mathbb{1}_{\{S-a\leq u\}}],\ \ \forall j \in \{1,\dots,d\},$$

where $a = \sum_{k=1}^d a_k$. Then, $(u_1^*,\dots,u_d^*)$ satisfies also:

$$\mathbb{E}[g_i'(u_i^* + a_i - X_i)\mathbb{1}_{\{X_i > u_i^* + a_i\}}\mathbb{1}_{\{S\leq u+a\}}] = \mathbb{E}[g_i'(u_i^* + a_j - X_j)\mathbb{1}_{\{X_j > u_j^* + a_i\}}\mathbb{1}_{\{S\leq u+a\}}],\ \ \forall j \in \{1,\dots,d\}.$$

Since, $(u_1^* + a_1,\dots,u_d^* + a_d) \in \mathcal{U}^d_{u+a}$, and from the solution uniqueness of the optimality condition (1.1) for the allocation $A_{X_1,\dots,X_d}(u+a)$, we deduce that: $u_k^* + a_k = u_k$ for all $k \in \{1,\dots,d\}$. □

Proposition 2.9 (Continuity)**.** *Under Assumptions H1 and H2, and if* $\forall k \in \{1,\dots,d\}$*,* $\exists \epsilon_0 > 0$ *such that:*

$$\forall \epsilon, |\epsilon| < \epsilon_0,\quad \mathbb{E}[\sup_{v\in[0,u]} |g_k'(v - (1+\epsilon)X_k)|] < +\infty,$$

then, if $(X_1,\dots,X_d)$ *is a continuous random vector, for all* $i \in \{1,\dots,d\}$*:*

$$\lim_{\epsilon\to 0} A_{X_1,\dots,(1+\epsilon)X_i,\dots,X_d}(u) = A_{X_1,\dots,X_i,\dots,X_d}(u).$$

Proof. Let $(u_1,\dots,u_d)$ be the optimal allocation of u on the d risks $(X_1,\dots,X_d)$:

$$(u_1,\dots,u_d) = A_{X_1,\dots,X_i,\dots,X_d}(u),$$

then $(u_1,\dots,u_d)$ is the unique solution in $\mathcal{U}^d_u$ of Equation system (1.1):

$$\mathbb{E}[g_i'(u_i - X_i)\mathbb{1}_{\{X_i > u_i\}}\mathbb{1}_{\{S\leq u\}}] = \mathbb{E}[g_i'(u_j - X_j)\mathbb{1}_{\{X_j > u_j\}}\mathbb{1}_{\{S\leq u\}}],\ \ \forall j \in \{1,\dots,d\}.$$

For $\epsilon \in \mathbb{R}$, let $(u_1^\epsilon,\dots,u_d^\epsilon)$ be the optimal allocation of u on the d risks $(X_1,\dots,X_{i-1},(1+\epsilon)X_i,X_{i+1},\dots,X_d)$:

$$(u_1^\epsilon,\dots,u_d^\epsilon) = A_{X_1,\dots,X_{i-1},(1+\epsilon)X_i,X_{i+1},\dots,X_d}(u),$$

then $(u_1^\epsilon, \ldots, u_d^\epsilon)$ is the unique solution in $\mathcal{U}_u^d$ of the following equation system:

$$\mathbb{E}[g_i'(u_i^\epsilon - (1+\epsilon)X_i)\mathbb{1}_{\{(1+\epsilon)X_i > u_i^\epsilon\}}\mathbb{1}_{\{S+\epsilon X_i \le u\}}] = \mathbb{E}[g_i'(u_j^\epsilon - X_j)\mathbb{1}_{\{X_j > u_j^\epsilon\}}\mathbb{1}_{\{S+\epsilon X_i \le u\}}], \quad \forall j \in \{1, \ldots, d\}.$$

Since $\mathcal{U}_u^d$ is a compact on $(\mathbb{R}^+)^d$, we may consider a convergent subsequence $(u_1^{\epsilon_k}, \ldots, u_d^{\epsilon_k})$ of $(u_1^\epsilon, \ldots, u_d^\epsilon)$.
Since the penalties functions satisfy:

$$\exists \epsilon_0 > 0, \quad \forall \epsilon, |\epsilon| < \epsilon_0, \quad \mathbb{E}[\sup_{v \in [0,u]} |g_k'(v - (1+\epsilon)X_k)|] < +\infty,$$

we use Lebesgue's dominated convergence Theorem to get:

$$\mathbb{E}[g_i'(\lim_{\epsilon \to 0} u_i^{\epsilon_k} - X_i)\mathbb{1}_{\{X_i > \lim_{\epsilon \to 0} u_i^{\epsilon_k}\}}\mathbb{1}_{\{S \le u\}}] = \mathbb{E}[g_i'(\lim_{\epsilon \to 0} u_j^{\epsilon_k} - X_j)\mathbb{1}_{\{X_j > \lim_{\epsilon \to 0} u_j^{\epsilon_k}\}}\mathbb{1}_{\{S \le u\}}], \quad \forall j \in \{1, \ldots, d\},$$

thereby $(\lim_{\epsilon \to 0} u_1^{\epsilon_k}, \ldots, \lim_{\epsilon \to 0} u_d^{\epsilon_k})$ is a solution of Equation (1.1), because $\sum_{l=1}^d \lim_{\epsilon \to 0} u_l^{\epsilon_k} = \lim_{\epsilon \to 0} \sum_{l=1}^d u_l^\epsilon = u$, $(\lim_{\epsilon \to 0} u_1^{\epsilon_k}, \ldots, \lim_{\epsilon \to 0} u_d^{\epsilon_k}) \in \mathcal{U}_u^d$.
From the solution uniqueness of (1.1) in $\mathcal{U}_u^d$, we deduce that: $\lim_{k \to \infty} u_i^{\epsilon_k} = u_i$ for all $i \in \{1, \ldots, d\}$.
For all convergent subsequence of $(u_1^\epsilon, \ldots, u_d^\epsilon)$ the limit point is $(u_1, \ldots, u_d)$, we deduce that:

$$\lim_{\epsilon \to 0}(u_1^\epsilon, \ldots, u_d^\epsilon) = (u_1, \ldots, u_d).$$

□

Proposition 2.10 (Monotonicity)**.** *Under Assumption H2, and for $(i,j) \in \{1, \ldots, d\}^2$, such that $g_i = g_j = g$:*

$$X_i \le_{st} X_j \Rightarrow u_i \le u_j.$$

Proof. Let $(u_1, \ldots, u_j)$ be the optimal allocation $A_{X_1,\ldots,X_i,\ldots,X_d}(u)$, under Assumption H2, the optimality condition (1.1) is written as follows:

$$\mathbb{E}[g'(u_i - X_i)\mathbb{1}_{\{X_i > u_i\}}\mathbb{1}_{\{S \le u\}}] = \mathbb{E}[g'(u_j - X_j)\mathbb{1}_{\{X_j > u_j\}}\mathbb{1}_{\{S \le u\}}].$$

Now if $X_i \le_{st} X_j$, and since, $x \mapsto -g'(-(u_i - x)_+)\mathbb{1}_{\{S \le u\}}$ is an increasing function on $\mathbb{R}^+$, then:

$$\mathbb{E}[g'(u_i - X_j)\mathbb{1}_{\{X_j > u_j\}}\mathbb{1}_{\{S \le u\}}] \le \mathbb{E}[g'(u_i - X_i)\mathbb{1}_{\{X_i > u_i\}}\mathbb{1}_{\{S \le u\}}].$$

We deduce that:

$$\mathbb{E}[g'(u_i - X_j)\mathbb{1}_{\{X_j > u_j\}}\mathbb{1}_{\{S \le u\}}] \le \mathbb{E}[g'(u_j - X_j)\mathbb{1}_{\{X_j > u_j\}}\mathbb{1}_{\{S \le u\}}],$$

and since, g' is an increasing function, and the distributions are all continuous, that implies: $u_j \ge u_i$. □

By combining all the properties demonstrated in this section, we show Theorem 1.7. These properties are therefore desirable from an economic point of view, the fact that they are satisfied by the proposed optimal allocation implies that this allocation method may well be used for the economic capital allocation between the different branches of a group, in terms of their actual participation in the overall risk, taking into account both their marginal distributions and their dependency structures with the remaining branches.

3. Some general results in independence case

In this section we generalize the results presented in dimension 2 by Cénac et al. in the first section of their paper [9] to higher dimension.
The results presented here give explicit forms to the optimal allocation for some specific distributions. This could be used as a benchmark to test optimization algorithms convergence.

We also get some asymptotic results, when the capital u goes to infinity. We study both the exponential and the sub-exponential cases, and we determine the difference between their asymptotic behavior for exponential and Pareto distributions cases.

We consider from now on that the penalty functions are identical and equal to the severity of local ruin $g_k(x) = g(x) = |x|, \forall k \in \{1, 2, \ldots, d\}$. The optimality conditions for minimizing the multivariate risk indicators I and J are given respectively by Equations (1.2) and (1.3). In this section we focus on the independence case.

Remark that $\alpha_i = \frac{u_i}{u} \in [0, 1]$, then when $u \to +\infty$, we may consider convergent subsequences in the proofs below. By abuse of notation, we consider $\lim\limits_{u\to+\infty} \alpha_i$. In fact, we consider a convergent subsequence and get the existence of the limit by obtaining the uniqueness of the limit point.

3.1. **Independent exponentials.**

Assume $X_1, X_2, \ldots, X_d$ are independent exponential random variables with respective parameters $0 < \beta_1 < \beta_2 < \cdots < \beta_d$. Remark that in the particular case where $\beta_1 = \beta_2 = \cdots = \beta_d$, the optimal allocation is $u_1 = \cdots = u_d = u/d$.

Proposition 3.1 (The optimal allocation for the indicator I)**.** *The allocation minimizing the risk indicator I is the unique solution in $\mathcal{U}_u^d$, of the following equation system :*

(3.1)
$$h(\beta_i\alpha_i) - h(\beta_j\alpha_j) - \sum_{l=1}^{d} A_l h(\beta_l)[h(\alpha_i \cdot (\beta_i - \beta_l)) - h(\alpha_j \cdot (\beta_j - \beta_l))] = 0, \forall (i, j) \in \{1, 2, \ldots, d\}^2,$$

where h is the function defined by $h(x) = \exp(-u \cdot x)$, and A_l denotes the constants $A_l = \prod\limits_{j=1, j\neq l}^{d} \dfrac{\beta_j}{\beta_j - \beta_l}$, $l = 1, \ldots, d$.

Proof. If $X_i \sim \mathcal{E}(\beta_i)$ are independent exponential random variables, then $S^{-i} = \sum\limits_{j=1; j\neq i}^{d} X_j$ have a generalized Erlang distribution with parameters $(\beta_1, \beta_2, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_d)$, so we write:

$$\begin{aligned}
\mathbb{P}\left(X_i > u_i, \sum_{j=1}^{d} X_j \leq u\right) &= \mathbb{P}\left(X_i > u_i\right) - \mathbb{P}\left(X_i > u_i, \sum_{j=1}^{d} X_j > u\right) \\
&= \bar{F}_{X_i}(u_i) - \bar{F}_{X_i}(u) - \int_{u_i}^{u} \bar{F}_{S^{-i}}(u - s) f_{X_i}(s) \mathrm{ds} \\
&= h(\beta_i\alpha_i) - h(\beta_i) - \sum_{l=1}^{d} A_l h(\beta_l) h(\alpha_i \cdot (\beta_i - \beta_l)) + \sum_{l=1}^{d} A_l h(\beta_i) \\
&= h(\beta_i\alpha_i) - \sum_{l=1}^{d} A_l h(\beta_l) h(\alpha_i \cdot (\beta_i - \beta_l)),
\end{aligned}$$

because, $\bar{F}_{X_i}(x) = e^{-\beta_i x}$, $\bar{F}_{S^{-i}}(x) = \sum_{l=1,l\neq i}^{d} (\prod_{j=1,j\neq l,j\neq i}^{d} \frac{\beta_j}{\beta_j - \beta_l})e^{-\beta_l x}$ and $\sum_{l=1}^{d} A_l = 1$.
The survival function of the generalized Erlang distribution with parameters $(\beta_1, \beta_2, \dots, \beta_d)$ is given by ([18]):

$$\bar{F}_X(x) = \sum_{l=1}^{d} (\prod_{j=1,j\neq l}^{d} \frac{\beta_j}{\beta_j - \beta_l})e^{-\beta_l x} = \sum_{l=1}^{d} A_l e^{-\beta_l x}.$$

The optimal allocation is the unique solution in $\mathcal{U}_u^d$, of the following equation system :

$$\mathbb{P}\left(X_i > u_i, \sum_{k=1}^{d} X_k \leq u\right) = \mathbb{P}\left(X_j > u_j, \sum_{j=k}^{d} X_k \leq u\right), \forall (i,j) \in \{1,2,\dots,d\}^2,$$

which leads to (3.1). ☐

The resulting system is a system of nonlinear equations, which can be solved numerically.

Proposition 3.2 (The asymptotic optimal allocation for the indicator I)**.** *When the capital u goes to infinity, the asymptotic optimal allocation satisfies:*

$$\lim_{u\to\infty} \alpha^* = \lim_{u\to\infty} \left(\frac{u_1}{u}, \frac{u_2}{u}, \dots, \frac{u_d}{u}\right) = \left(\frac{\frac{1}{\beta_i}}{\sum_{j=1}^{d} \frac{1}{\beta_j}}\right)_{i=1,2,\dots,d}.$$

Proof. Equation system (3.1) is equivalent to:

(3.2) $$\forall (i,j) \in \{1,2,\dots,d\}^2, \quad h(\beta_i\alpha_i)[1 - \sum_{l=1}^{d} A_l h((1-\alpha_i)\beta_l)] = h(\beta_j\alpha_j)[1 - \sum_{l=1}^{d} A_l h((1-\alpha_j)\beta_l)].$$

Firstly, remark that for all $i \in \{1,2,\dots,d\}$, $\limsup_{u\to\infty} \frac{u_i}{u} < 1$ because, if this result was not satisfied then taking if necessary i such that $\lim_{u\to\infty} \frac{u_i}{u} = \lim_{u\to\infty} \alpha_i = 1$. Then for all $j \neq i$, $\lim_{u\to\infty} \frac{u_j}{u} = \lim_{u\to\infty} \alpha_j = 0$, and Equation system (1.2) cannot be satisfied.
Equation system (3.2) is equivalent to:

$$\forall (i,j) \in \{1,2,\dots,d\}^2, \quad h(\beta_i\alpha_i - \beta_j\alpha_j) = \frac{1 - \sum_{l=1}^{d} A_l h((1-\alpha_j)\beta_l)}{1 - \sum_{l=1}^{d} A_l h((1-\alpha_i)\beta_l)},$$

the right side of the last system equations tends to 1 when u tends to ∞, therefore, we deduce that $\lim_{u\to\infty} h(\beta_i\alpha_i - \beta_j\alpha_j) = 1$ and consequently:

$$\forall (i,j) \in \{1,2,\dots,d\}^2, \lim_{u\to\infty} \alpha_i = \frac{\beta_j}{\beta_i} \lim_{u\to\infty} \alpha_j,$$

then, for all $i \in \{1,2,\dots,d\}$:

$$\lim_{u\to\infty} \alpha_i = \frac{\frac{1}{\beta_i}}{\sum_{j=1}^{d} \frac{1}{\beta_j}}.$$

☐

Remark 3.3. Based on the above result, we can conclude that asymptotically:

- if $\beta_i < \beta_j$ then $\alpha_i > \alpha_j$, this means that we allocate more capital to the most risky business line.

- α_i is a decreasing function of β_i. This observation is consistent with the previous conclusion.
- α_j for $j \neq i$ is an increasing function of β_i.

Proposition 3.4 (The optimal allocation for the indicator J)**.** *The allocation minimizing the risk indicator J is the unique solution in $\mathcal{U}_u^d$, of the following equation system :*

$$\forall (i,j) \in \{1,2,\ldots,d\}^2, \quad \sum_{l=1}^{d} A_l h(\beta_l)[h(\alpha_i \cdot (\beta_i - \beta_l)) - h(\alpha_j \cdot (\beta_j - \beta_l))] = 0. \tag{3.3}$$

Proof. The proof is similar to that of Proposition 3.1. □

Proposition 3.5 (The asymptotic optimal allocation for the indicator J)**.** *When the capital u goes to infinity, the optimal allocation minimizing the risk indicator J is the following:*

$$\lim_{u\to\infty} \frac{u_1}{u} = 1 \text{ and } \lim_{u\to\infty} \frac{u_j}{u} = 0 \ \forall j \in \{2,3,\ldots,d\}.$$

Proof. Equation system (3.3) is equivalent to the following one:

$$\forall (i,j) \in \{1,2,\ldots,d\}^2, \quad \sum_{l=1}^{d} A_l \cdot h((1-\alpha_i)\cdot(\beta_l - \beta_i)) = \sum_{l=1}^{d} A_l \cdot h(\alpha_j \cdot (\beta_j - \beta_l) + \beta_l - \beta_i).$$

If $\overline{\lim}\, \frac{u_1}{u} < 1$, as u goes to $+\infty$ in the equations of the previous system for $(i = 1)$ we get:

$$\forall j \in \{2,3,\ldots,d\}, \quad \lim_{u\to\infty} \sum_{l=1}^{d} A_l \cdot h(\alpha_j \cdot (\beta_j - \beta_l) + \beta_l - \beta_1) = A_1.$$

The first terms of these equations can be decomposed into three parts as follows:

$$\begin{aligned}\lim_{u\to\infty} \sum_{l=1}^{d} A_l \cdot h(\alpha_j \cdot (\beta_j - \beta_l) + \beta_l - \beta_1) = & \lim_{u\to\infty} A_1 \cdot h(\alpha_j \cdot (\beta_j - \beta_1)) \\ & + \lim_{u\to\infty} \sum_{l=2}^{j} A_l \cdot h(\alpha_j \cdot (\beta_j - \beta_l) + \beta_l - \beta_1) \\ & + \lim_{u\to\infty} \sum_{l=j+1}^{d} A_l \cdot h(\alpha_j \cdot (\beta_j - \beta_l) + \beta_l - \beta_1),\end{aligned}$$

and since, for all, $j > 1, \lim_{u\to\infty} \sum_{l=2}^{j} A_l \cdot h(\alpha_j \cdot (\beta_j - \beta_l) + \beta_l - \beta_1) = 0$, because $\beta_l - \beta_1 > 0$ and $\beta_j - \beta_l \geq 0$ for $l \in \{2,3,\ldots,j\}$. Moreover, $\lim_{u\to\infty} \sum_{l=j+1}^{d} A_l \cdot h(\alpha_j \cdot (\beta_j - \beta_l) + \beta_l - \beta_1) = 0$, because for all $l \in \{j+1, j+2, \ldots, d\}$, $\alpha_j \cdot (\beta_j - \beta_l) + \beta_l - \beta_1 = (\beta_l - \beta_j)(1-\alpha_j) + \beta_j - \beta_1 > 0$, then $\lim_{u\to\infty} h(\alpha_j \cdot (\beta_j - \beta_1)) = 1$. We deduce that $\forall j \in \{2,3,\ldots,d\}$; $\lim_{u\to\infty} \alpha_j = o(\frac{1}{u})$. This contradicts the necessary condition: $\lim_{u\to\infty} \sum_{l=1}^{d} \alpha_j = 1$. □

3.2. **Some distributions of the sub-exponential family.**

In most cases of risk distributions, we cannot give explicit or semi-explicit optimal allocation formulas, the difficulty comes from the lack of a simple form of the risks sum and its joint distribution with each risk. In this section, we present explicit formulas for some distributions of the sub-exponential family. By this way, we generalize results of [9] to higher dimension.

We recall the sub-exponential distributions family definition, consisting in distributions of positive support, with an distribution function that satisfies:

$$\frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \overset{x\to+\infty}{\longrightarrow} 2,$$

where $\overline{F^{*2}}$ is the convolution of $\overline{F}$.
In Asmussen (2000) [2], it is proven that the sub-exponential distributions satisfy also the following relation, for all $d \in \mathbb{N}^*$:

$$\frac{\overline{F^{*d}}(x)}{\overline{F}(x)} \overset{x\to+\infty}{\longrightarrow} d,$$

where $\overline{F^{*d}}$ is the d^{th}convolution of $\overline{F}$.
We shall use the following theorem proved in Cénac and al.

Theorem 3.6 (Sub-exponential distributions [9])**.** *Let X be a random variable with sub-exponential distribution F_X, Y a random variable with support $\mathbb{R}^+$, independent of X, and $(u,v) \in (\mathbb{R}^+)^2$, such that:*

- *there exists $0 < \kappa_1 < \kappa_2 < 1$ such that for u large enough, $\kappa_1 \leq \frac{v}{u} \leq \kappa_2$,*
- *$\frac{\bar{F}_X(y)}{\bar{F}_X(x)} \overset{x\to+\infty}{=} O(1)$, if $y = \Theta(x)$*[1]*.*

Then,

$$\lim_{u\to\infty} \frac{\mathbb{P}(X \geq v,\ X+Y \geq u)}{\bar{F}_X(u)} = 1.$$

3.2.1. *The asymptotic behavior.*
Propositions 3.8 and 3.10 concern the asymptotic behavior of the optimal allocation by minimizing the indicator I, in the cases of some sub-exponential distributions.
In what follows, $(u_1, \ldots, u_d)$ denotes the optimal allocation of u associated to the risk indicator I.

Theorem 3.7. *Let $(X_1, X_2, \ldots, X_d)$ be continuous positive and independent random variables. Assume that $\forall (i,j) \in \{1,2,\ldots,d\}^2$:*

(1) $\bar{F}_{X_i}(x) \overset{x\to+\infty}{=} \Theta(\bar{F}_{X_j}(x))$,
(2) $\bar{F}_{X_i}(s) \overset{s\to+\infty}{=} o(\bar{F}_{X_i}(t))$, *if $t = o(s)$,*

then, there exist $\kappa_1 > 0$ and $\kappa_2 < 1$ such that,

$$\kappa_1 \leq \frac{u_l}{u} \leq \kappa_2 \quad \forall l \in \{1,2,\ldots,d\}. \tag{3.4}$$

Note that the first condition of Theorem 3.7 is not satisfied for exponential distributions. However, Pareto distributions verify the hypothesis of Theorem 3.7.

Proof. We suppose that, $\exists i \in \{1,\ldots,d\}$ such that: $\frac{u_i}{u} \overset{u\to+\infty}{\longrightarrow} 1$ or $\frac{u_i}{u} \overset{u\to+\infty}{\longrightarrow} 0$, the first case implies that foll all $j \neq i$, $\frac{u_j}{u} \overset{u\to+\infty}{\longrightarrow} 0$, then, it is sufficient to prove that the existence of an $i \in \{1,\ldots,d\}$ such that $\frac{u_i}{u} \overset{u\to+\infty}{\longrightarrow} 0$ is impossible.
We assume the existence of $i \in \{1,\ldots,d\}$, such that: $\frac{u_i}{u} \overset{u\to+\infty}{\longrightarrow} 0$.

[1]For $(x,y) \in \mathbb{R}^{2+}$, we shall denote $y = \Theta(x)$ if there exist $0 < C_1 \leq C_2 < \infty$, such that for x large enough, $C_1 \leq \frac{y}{x} \leq C_2$

Then, $\exists j \in \{1,\ldots,d\}\backslash i$ such that $\lim\limits_{u\to+\infty} \frac{u_j}{u} \in]0,1]$, therefore, $u_j \overset{u\to+\infty}{\longrightarrow} +\infty$ and $u_i \overset{u\to+\infty}{=} o(u_j)$.
Using Assumptions (1) and (2), we deduce that:

$$\frac{\bar{F}_{X_j}(u_j)}{\bar{F}_{X_i}(u_i)} \overset{u\to+\infty}{\longrightarrow} 0. \tag{3.5}$$

The optimality condition ((1.2)) can also be written for all $j \neq i$ as follows:

$$\bar{F}_{X_i}(u_i) - \mathbb{P}(X_i > u_i, S > u) = \bar{F}_{X_j}(u_j) - \mathbb{P}(X_i > u_j, S > u). \tag{3.6}$$

That presents a trivial contradiction if u_i remains bounded.
Now, assume that $u_i \to +\infty$. Recall that $S^{-i} = \sum\limits_{k=1,k\neq i}^{k=d} X_k$, then:

$$\mathbb{P}(X_i > u_i, S > u) = \mathbb{P}\left(X_i > u_i, S^{-i} > \sqrt{u}, S > u\right) + \mathbb{P}\left(X_i > u_i, S^{-i} < \sqrt{u}, S > u\right).$$

We have:

$$\mathbb{P}\left(X_i > u_i, S^{-i} > \sqrt{u}, S > u\right) \leq \mathbb{P}(X_i > u_i)\,\mathbb{P}\left(S^{-i} > \sqrt{u}\right) \overset{u\to+\infty}{=} o(\bar{F}_{X_i}(u_i)).$$

Using assumption (2) and since $u_i = o(u)$,

$$\mathbb{P}\left(X_i > u_i, S^{-i} < \sqrt{u}, S > u\right) \leq \bar{F}_{X_i}(u - \sqrt{u}) \overset{u\to+\infty}{=} o(\bar{F}_{X_i}(u_i)).$$

We deduce that:

$$\mathbb{P}(X_i > u_i, S > u) \overset{u\to+\infty}{=} o(\bar{F}_{X_i}(u_i)). \tag{3.7}$$

We remark also that:

$$\mathbb{P}(X_j > u_j, S > u) \overset{u\to+\infty}{=} O(\bar{F}_{X_j}(u_j)) \overset{u\to+\infty}{=} o(\bar{F}_{X_i}(u_i)). \tag{3.8}$$

Equation (3.6) leads to:

$$1 - \underbrace{\frac{\mathbb{P}(X_i > u_i, S > u)}{\bar{F}_{X_i}(u_i)}}_{T_1} = \underbrace{\frac{\bar{F}_{X_j}(u_j)}{\bar{F}_{X_i}(u_i)}}_{T_2} - \underbrace{\frac{\mathbb{P}(X_j > u_j, S > u)}{\bar{F}_{X_i}(u_i)}}_{T_3}.$$

Now, relations: (3.7), (3.5), and (3.8), imply that T_1, T_2, and T_3, go to zero, and this is a contradiction. $\square$

Proposition 3.8. *Let $X_1,\ldots,X_d$ be continuous, positive and independent random variables such that the support of the density of (X_i, S) is $(\mathbb{R}^+)^2$. Let $(u_1,\ldots,u_d)$ be the optimal allocation of u associated to the risk indicator I. We assume:*

(1) *there exist $0 < \kappa_1 < \kappa_2 < 1$ such that for all $i = 1,\ldots,d$ and for all $u \in \mathbb{R}^+$,*

$$\kappa_1 \leq \frac{u_i}{u} \leq \kappa_2,$$

(2) *for all $i = 1,\ldots,d$, if $y = y(x)$ is such that*

$$0 < \liminf_{x\to\infty} \frac{y}{x} \leq \limsup_{x\to\infty} \frac{y}{x} < 1$$

then

$$\frac{\overline{F}_{X_i}(x)}{\overline{F}_{X_i}(y)} \overset{x\to\infty}{\longrightarrow} 0.$$

Then, for all $i, j = 1,\ldots,d$,

$$\frac{\overline{F}_{X_i}(u_i)}{\overline{F}_{X_j}(u_j)} \overset{u\to\infty}{\longrightarrow} 1.$$

Assumptions of Proposition 3.8 are satisfied for distributions of exponential type (see remark (3.9) below). Its application gives another proof to Proposition 3.2. In contrast, Proposition 3.8 cannot be used for Pareto distributions.

Proof. Following the lines of the proof of Theorem 3.7, take $0 < \gamma < 1 - \kappa_2$,

$$\mathbb{P}(X_i > u_i,\ S > u) = \mathbb{P}(X_i > u_i,\ S^{-i} > \gamma u,\ S > u) + \mathbb{P}(X_i > u_i,\ S^{-i} > \gamma u,\ S > u).$$

As before,

$$\mathbb{P}(X_i > u_i,\ S^{-i} > \gamma u,\ S > u) \leq \mathbb{P}(X_i > u_i)\mathbb{P}(S^{-i} > \gamma u) \overset{u \to +\infty}{=} o(\overline{F}_{X_i}(u_i)).$$

On the other hand,

$$\begin{aligned}
\mathbb{P}(X_i > u_i,\ S^{-i} > \gamma u,\ S > u) &\leq \mathbb{P}(X_i > u - \gamma u) \\
&= \overline{F}_{X_i}((1-\gamma)u) \overset{u \to +\infty}{=} o(\overline{F}_{X_i}(u_i)) \\
&\quad \text{because } 0 < \frac{\kappa_1}{1-\gamma} \leq \frac{u_i}{(1-\gamma)u} \leq \frac{\kappa_2}{1-\gamma} < 1.
\end{aligned}$$

So that, $\mathbb{P}(X_i > u_i,\ S > u) \overset{u \to +\infty}{=} o(\overline{F}_{X_i}(u_i))$ and the same computation gives $\mathbb{P}(X_j > u_j,\ S > u) \overset{u \to +\infty}{=} o(\overline{F}_{X_j}(u_j))$. Now, u_i and u_j satisfy Equation (1) and thus,

$$1 + o(1) \overset{u \to +\infty}{=} \frac{\overline{F}_{X_j}(u_j)}{\overline{F}_{X_i}(u_i)} + o(1)\frac{\overline{F}_{X_j}(u_j)}{\overline{F}_{X_i}(u_i)}.$$

This implies that $\frac{\overline{F}_{X_j}(u_j)}{\overline{F}_{X_i}(u_i)}$ is bounded from above and thus

$$\frac{\overline{F}_{X_j}(u_j)}{\overline{F}_{X_i}(u_i)} \overset{u \to +\infty}{=} 1 + o(1).$$

□

Remark 3.9. We remark that the hypothesis of Proposition 3.8 are satisfied for distribution of *exponential type*, that is distributions verifying:

$$\bar{F}_{X_i}(x) = \Theta(e^{-\mu_i x}).$$

Indeed, in this case, we have $0 < \underline{\lim}\, \frac{u_i}{u} \leq \overline{\lim}\, \frac{u_i}{u} < 1, \quad \forall i \in \{1, \dots, d\}$. In fact, if $u_i \overset{u \to +\infty}{=} o(u)$, then, $\exists j \in \{1, \dots, d\} \backslash \{i\}$ such that $\frac{u_j}{u} \to \kappa \in]0,1]$, so $u_i \overset{u \to +\infty}{=} o(u_j)$.
Since $\mu_i, \mu_j > 0$, $\mu_i u_i \overset{u \to +\infty}{=} o(\mu_j u_j)$. And as in the proof of Theorem 3.7, we get:

$$\mathbb{P}\left(X_i > u_i, S > u\right) \overset{u \to +\infty}{=} o(\bar{F}_{X_i}(u_i))$$

and,

$$\mathbb{P}\left(X_j > u_j, S > u\right) \overset{u \to +\infty}{=} o(\bar{F}_{X_j}(u_j)).$$

From the optimality condition,

$$\frac{\bar{F}_{X_i}(u_i)}{e^{-\mu_i u_i}} - \underbrace{\frac{\mathbb{P}\left(X_i > u_i, S > u\right)}{e^{-\mu_i u_i}}}_{T_1} = \underbrace{\frac{\bar{F}_{X_j}(u_j)}{e^{-\mu_i u_i}}}_{T_2} - \underbrace{\frac{\mathbb{P}\left(X_j > u_j, S > u\right)}{e^{-\mu_i u_i}}}_{T_3},$$

which is absurd because as $u \to +\infty$:

- $\bar{F}_{X_i}(u_i) = \Theta(e^{-\mu_i u_i})$,
- $T_1 = o(1)$,
- $T_2 = \frac{\bar{F}_{X_j}(u_j)}{e^{-\mu_j u_j}} e^{-\mu_j u_j + \mu_i u_i} \to 0$, since $\mu_i u_i = o(\mu_j u_j)$,

- and $T_3 = o(1)e^{-\mu_j u_j + \mu_i u_i} \to 0$.

Proposition 3.10 (The asymptotic optimal allocation for the indicator I)**.** *Under the same conditions of Theorem 3.7, and if, for all $i \in \{1, \ldots, d\}$, F_{X_i} is a sub-exponential distribution, that verifies:*

$$\frac{\bar{F}_{X_i}(y)}{\bar{F}_{X_i}(x)} \overset{x \to +\infty}{=} O(1), \quad \textit{for} \ \ 0 < \kappa_1 \leq \frac{y}{x} \leq \kappa_2 < 1.$$

Then, by minimizing the I indicator, u_i and u_j satisfy:

$$\bar{F}_{X_i}(u_i) - \bar{F}_{X_i}(u) \overset{u \to +\infty}{=} \bar{F}_{X_j}(u_j) - \bar{F}_{X_j}(u) + o(\bar{F}_{X_i}(u)). \tag{3.9}$$

Proposition 3.10 is applicable in the case of Pareto distributions. So, we will use it for determining the optimal asymptotic allocation, for independent risks of Pareto distributions in the next subsection.

Proof. The proof of this theorem is a direct application of Theorems 3.6 and 3.7. □

Now, we focus on the asymptotic optimal allocation by minimizing the risk indicator J, and we study the case of sub-exponential distribution family.

Proposition 3.11 (The asymptotic optimal allocation for the indicator J)**.** *Let $(X_1, X_2, \ldots, X_d)$ be continuous positive and independent random variables, such that, there exist $i \in \{1, \ldots, d\}$ with a sub-exponential distribution, the optimal capital allocation by minimizing the J indicator $(u_1, \ldots, u_d)$ verifies, for all $j \neq i$:*

$$\lim_{u \to \infty} \frac{\mathbb{P}(X_j \geq u_j, \ S \geq u)}{\bar{F}_{X_i}(u)} = 1.$$

Proof. The solution to (1.3) satisfies :

$$\forall j \in \{1, 2, \ldots, d\}, \quad \frac{\mathbb{P}\left(X_i > u_i, S \geq u\right)}{\mathbb{P}\left(X_i > u\right)} = \frac{\mathbb{P}\left(X_j > u_j, S \geq u)\right)}{\mathbb{P}\left(X_i > u\right)}.$$

When u goes to $+\infty$, and using Theorem 3.6, we obtain Proposition 3.11. □

3.2.2. *application to Pareto independent distributions.*
We consider d independent random variables $\{X_1, X_2, \ldots, X_d\}$ of Pareto distributions, with parameters $(a, b_i)_{\{i=1,2,\ldots,d\}}$ respectively, such that $b_1 > b_2 > \cdots > b_d > 0$. Therefore, these distributions will be characterized by densities and survival functions of the following forms:

$$f_{X_i}(x) = \frac{a}{b_i}\left(1 + \frac{x}{b_i}\right)^{-a-1},$$

and

$$\bar{F}_{X_i}(x) = \left(1 + \frac{x}{b_i}\right)^{-a}.$$

Proposition 3.12 (The asymptotic optimal allocation minimizing I)**.** *Asymptotically, the unique solution to (1.2) satisfies:*

$$\forall (i,j) \in \{1, 2, \ldots, d\}^2, \qquad \left(\frac{\lim\limits_{u \to \infty} \alpha_i}{b_i}\right)^{-a} - \left(\frac{\lim\limits_{u \to \infty} \alpha_j}{b_j}\right)^{-a} = \left(\frac{1}{b_i}\right)^{-a} - \left(\frac{1}{b_j}\right)^{-a}.$$

Proof. Follows from Proposition 3.10. □

Proposition 3.13 (The asymptotic optimal allocation minimizing J)**.** *The unique solution to (1.3) satisfies:*

$$\lim_{u\to\infty}\alpha_1 = 1 \quad et \quad \lim_{u\to\infty}\alpha_i = 0, \forall i \in \{2,3,\dots,d\}.$$

Proof. We suppose that, $\exists \in \{1,\dots,d\}$ such that $0 < \lim_{u\to\infty}\alpha_j < 1$.
From Theorem 3.6 :

$$\lim_{u\to\infty}\frac{\mathbb{P}(X_j \geq u_j, S \geq u)}{\bar{F}_{X_j}(u)} = 1.$$

On the other hand, and applying Proposition 3.11 in the Pareto distributions case, we get for $i \in \{1,\dots,d\}\setminus\{j\}$:

$$\frac{\mathbb{P}(X_j \geq u_j, S \geq u)}{\bar{F}_{X_j}(u)} = \frac{\mathbb{P}(X_j \geq u_j, S \geq u)}{\bar{F}_{X_i}(u)} \cdot \frac{\bar{F}_{X_i}(u)}{\bar{F}_{X_j}(u)} \overset{u\to+\infty}{\sim} \frac{\bar{F}_{X_i}(u)}{\bar{F}_{X_j}(u)} = \left(\frac{1+\frac{u}{b_i}}{1+\frac{u}{b_j}}\right)^{-a},$$

then, for $i \in \{1,\dots,d\}\setminus\{j\}$:

$$\lim_{u\to\infty}\frac{\mathbb{P}(X_j \geq u_j, S \geq u)}{\bar{F}_{X_j}(u)} = \left(\frac{b_j}{b_i}\right)^{-a} \neq 1.$$

That is absurd. We deduce that $\forall i \in \{1,2,\dots,d\}$:

$$\lim_{u\to\infty}\alpha_i \in \{0,1\},$$

and since $\sum_{i=1}^d \alpha_i = 1$, then, there is a unique i such that $\lim_{u\to\infty}\alpha_i = 1$ and for all $j \neq i$ $\lim_{u\to\infty}\alpha_j = 0$. Now, and using the stochastic dominance order, we have $X_d \preccurlyeq_{st} \cdots \preccurlyeq_{st} X_2 \preccurlyeq_{st} X_1$, because $b_1 > b_2 > \cdots > b_d$. Finally, since the optimal capital allocation satisfies the monotonicity property, we deduce that:

$$\max_{i\in\{1,2,\dots,d\}}\{\lim_{u\to\infty}\alpha_i\} = \lim_{u\to\infty}\alpha_1 = 1.$$

□

3.3. **Comparison between the asymptotic behaviors in exponential and sub-exponential cases.** Let us consider the following risk indicator:

$$I_{loc}(u_1,\dots,u_d) = \sum_{k=1}^d \mathbb{E}\left((X_k - u_k)\mathbb{1}_{\{X_k>u_k\}}\right) = \sum_{k=1}^d \mathbb{E}\left((X_k-u_k)^+\right).$$

With this indicator, the impact of each branch on the group is not taken into account. This indicator thus takes only *local* effects into account. If, for any $k = 1,\dots,d$, the random vector (X_k, S) admits a density whose support contains $[0,u]^2$, then the optimality condition associated to I_{loc} gives:

$$\mathbb{P}(X_i > u_i) = \mathbb{P}(X_j > u_j).$$

We remark that this corresponds to the asymptotic result for indicator I of Proposition 3.8, where $\frac{\overline{F}_{X_j}(u_j)}{\overline{F}_{X_i}(u_i)} \overset{u\to+\infty}{=} 1 + o(1)$. In other words, in the case of *exponential type* and independent risks, the group effect is negligible. This behavior is also clear in Proposition 3.2, where we found the asymptotic capital allocation for I for independent exponential distributions as:

$$u_i = \frac{\frac{1}{\beta_i}}{\sum_{j=1}^d \frac{1}{\beta_j}} u, \text{ for all } i = 1,\dots,d,$$

as for I_{loc}.

Proposition 3.10 shows that for independent sub-exponential distributions the asymptotic behavior is different. Indeed, in Equation (3.9), the terms $\bar{F}_{X_i}(u)$ and $\bar{F}_{X_j}(u)$ lead to take into account the group effect. In the Pareto distributions case as example, recall that Proposition 3.12 gives that the asymptotic behavior of the optimal allocation for I is described by :

$$\left(\frac{\lim\limits_{u\to\infty}\alpha_i}{b_i}\right)^{-a}-\left(\frac{\lim\limits_{u\to\infty}\alpha_j}{b_j}\right)^{-a}=\left(\frac{1}{b_i}\right)^{-a}-\left(\frac{1}{b_j}\right)^{-a},$$

whereas the optimal allocation for I_{loc} for independent Pareto distributions is given by $\alpha_i=\frac{b_i}{\sum_{l=1}^{d}b_l}$, for all $i\in\{1,\dots,d\}$.

For optimal allocation by minimizing the J indicator, the asymptotic behavior is identical for the two families of distributions, the riskiest branch is considered as first responsible for the overall ruin, and thus, the optimal solution is to allocate the entire capital u to this business line.

4. The impact of the dependence structure

In this section, we focus on the impact of the dependence structure on the optimal allocation. We study at first the impact of mixture exponential-gamma to construct a Pareto distribution, compared to the independence case presented in the previous section, then we analyze the optimal allocation composition in the case of comonotonic risks. The last sub-section is devoted to the study of the impact of the dependence nature on the optimal allocation, using some bivariate models with copulas.

4.1. **Correlated Pareto.**

Let $(X_1,\dots,X_d)$ be a mixture of exponential distributions such that for all $i\in\{1,2,\dots,d\}$, $X_i\sim\mathcal{E}(\beta_i\theta)$, with $(\beta_1<\beta_2\cdots<\beta_d)$, and $\theta\sim\Gamma(a,b)$. Therefore, X_i have survival functions of the form:

$$\bar{F}_{X_i}(x)=\int_0^\infty \bar{F}_{X_i|\Theta=\theta}f_\Theta(\theta)d\theta=\int_0^\infty e^{-\beta_i\theta x}f_\Theta(\theta)d\theta=\left(1+\frac{\beta_i x}{b}\right)^{-a},$$

consequently, X_i have Pareto distribution of parameters $\left(a,\frac{b}{\beta_i}\right)$. They are conditionally independents. So, the idea is conditioning on the random variable θ and then integrate the formulas found for the case of independent exponential distributions. This model has been studied in e.g.[20, 26].

Proposition 4.1 (The optimal allocation for the indicator I)**.** *The optimal allocation minimizing the multivariate risk indicator I is the unique solution in $\mathcal{U}_u^d$, of the following equation system :*

$$\forall(i,j)\in\{1,2,\dots,d\}^2,$$

$$s(\beta_i\alpha_i)-s(\beta_j\alpha_j)-\sum_{l=1}^{d}A_l[s(\alpha_i\beta_i+(1-\alpha_i)\beta_l))-s(\alpha_j\beta_j+(1-\alpha_j)\beta_l))]=0, \tag{4.1}$$

where s is the function defined by $s(x)=(1+x\frac{u}{b})^{-a}$ and $\alpha_i=\frac{u_i}{u}$ for all $i\in\{1,\dots,d\}$.

Proof. It suffices to integrate Equation system (3.1), multiplied by the density function of θ. □

Proposition 4.2 (The asymptotic optimal allocation for the indicator I)**.** *When the capital u goes to infinity, the optimal allocation by minimization of the risk indicator I is the unique solution in $\mathcal{U}_u^d$ of the following equation system:*

$$\forall(i,j)\in\{1,2,\dots,d\}^2,$$

$$(4.2)\qquad (\beta_i\alpha_i)^{-a} - (\beta_j\alpha_j)^{-a} - \sum_{l=1}^{d} A_l[(\alpha_i\beta_i + (1-\alpha_i)\beta_l)^{-a} - (\alpha_j\beta_j + (1-\alpha_j)\beta_l)^{-a}] = 0.$$

Proof. We divide Equation system (4.1) by s(1), and let u go to $+\infty$ to get Equation system (4.2). $\square$

Proposition 4.2 shows the impact of the dependence related to the mixture. Indeed, in the case of independent Pareto distributions, of parameters $\left(a, \frac{b}{\beta_i}\right)_{i=1,\dots,d}$, the asymptotic optimal allocation for the indicator I is given by Proposition 3.12 as the solution of the equation system:

$$\forall (i,j) \in \{1,2,\dots,d\}^2, \qquad (\beta_i\alpha_i)^{-a} - (\beta_j\alpha_j)^{-a} = (\beta_i)^{-a} - (\beta_j)^{-a}.$$

Each equation in this system depends only on two risks, unlike the mixture case, where the equations of Equation system (4.2), depend on all the risks.

Proposition 4.3 (The optimal allocation for the indicator J)**.** *The optimal allocation minimizing the multivariate risk indicator J is the unique solution in $\mathcal{U}_u^d$, of the following equation system :*

$$(4.3)\qquad \forall (i,j) \in \{1,2,\dots,d\}^2, \sum_{l=1}^{d} A_l[s(\alpha_i\beta_i + (1-\alpha_i)\beta_l)) - s(\alpha_j\beta_j + (1-\alpha_j)\beta_l))] = 0.$$

Proof. It suffices to integrate Equation system (3.3), multiplied by the density function of θ. $\square$

Proposition 4.4 (The asymptotic optimal allocation for the indicator J)**.** *When the capital u goes to infinity, the optimal allocation by minimization of the risk indicator J, is the unique solution in $\mathcal{U}_u^d$ of the following equation system:*

$$(4.4)\qquad \forall (i,j) \in \{1,2,\dots,d\}^2, \quad \sum_{l=1}^{d} A_l[(\alpha_i\beta_i + (1-\alpha_i)\beta_l)^{-a} - (\alpha_j\beta_j + (1-\alpha_j)\beta_l)^{-a}] = 0.$$

Proof. we divide Equation system (4.3) by s(1), and we let u fo to $+\infty$ to get Equation system (4.4). $\square$

Proposition 4.4 shows that for the indicator J, the asymptotic behaviour of the optimal capital allocation takes into account the mixture effect. In fact, for independent Pareto distributions, we have proved in Proposition 3.13, that we allocate the entire capital u to the riskiest branch X_1, while the asymptotic optimal allocation in the correlated Pareto distributions case is the solution of Equation system (4.4).

4.2. **Comonotonic risks.**
A vector of random variables $(X_1, X_2, \dots, X_n)$ is comonotonic if and only if there exists a random variable Y and non decreasing functions $\varphi_1, \dots, \varphi_n$ such that:

$$(X_1, \dots, X_n) =_d (\varphi_1(Y), \dots, \varphi_n(Y)).$$

Note that the joint distribution function of comonotonic random variables corresponds to the upper Fréchet bound.
In the case where the risks X_1,…,X_d are comonotonic, we can give explicit formulas for the optimal allocation minimizing the multivariate risk indicators I and J, and for some risk models. For that, we use the existence of an uniform random variable U such that: $X_i = F_{X_i}^{-1}(U)$ for all $i \in \{1,\dots,d\}$, and $S = \sum_{i=1}^{d} F_{X_i}^{-1}(U) = \varphi(U)$, where $\varphi(t) = \sum_{i=1}^{d} F_{X_i}^{-1}(t)$, φ is a non decreasing function.
The main result of this section is given below.

Proposition 4.5. *Let $X_1,\ldots,X_d$ be comonotonic risks, with increasing distribution functions and support containing $[0,u]$. The optimal allocations for indicators I and J coincide, they are given by: $(u_1,\ldots,u_d)\in\mathcal{U}_u^d$ and*

$$F_{X_i}(u_i)=F_{X_j}(u_j)\ \forall\ i,j=1,\ldots,d.$$

Proof. Let us denote: $w_i=F_{X_i}(u_i)$, $v=\varphi^{-1}(u)$, $M_i=\max(w_i,v)$. The indicators I and J may be rewritten for $(u_1,\ldots,u_d)\in\mathcal{U}_u^d$:

$$\begin{aligned} I(u_1,\ldots,u_d) &= \sum_{i=1}^{d}\mathbb{E}\left((F_{X_i}^{-1}(U)-u_i)\mathbf{1}_{\{U\geq w_i,\ U\leq v\}}\right)\\ J(u_1,\ldots,u_d) &= \sum_{i=1}^{d}\mathbb{E}\left((F_{X_i}^{-1}(U)-u_i)\mathbf{1}_{\{U\geq M_i\}}\right). \end{aligned}$$

We remark that since $(u_1,\ldots,u_d)\in\mathcal{U}_u^d$, and F_{X_i} is strictly increasing for all $i\in\{1,\ldots,d\}$, we cannot have that $w_i<v$ for all i, so that, I is not trivially equal to 0. We use Lagrange multiplier to get that the minimum of I and J are reached in $\mathcal{U}_u^d$ respectively for:

- $\mathbb{P}(U\geq w_i,\ U\leq v)=\mathbb{P}(U\geq w_j,\ U\leq v)$, for $i,j=1,\ldots,d$,
- $\mathbb{P}(U\geq M_i)=\mathbb{P}(U\geq M_j)$, for $i,j=1,\ldots,d$.

These equality are acheaved if and only if $w_i=w_j=v$ or in other words if $F_{X_i}(u_i)=F_{X_j}(u_j)$. We remark that the minimum of I is then 0. □

The following three corollaries are direct applications of Proposition 4.5 to some particular cases.

Corollary 4.6 (Comonotonic exponential model)**.** *For comonotonic risks of exponential distributions $X_i\sim\exp(\beta_i)$, the optimal allocation by minimization of the two risk indicators is:*

$$\forall i\in\{1,\ldots,d\},\quad u_i=\frac{1/\beta_i}{\sum_{j=1}^{d}1/\beta_j}u.$$

Corollary 4.7 (Comonotonic log-normal model)**.** *For comonotonic risks of log-normal distributions $X_i\sim LN(\mu_i,\sigma)$, the optimal allocation by minimization of the two risk indicators is:*

$$\forall i\in\{1,\ldots,d\},\quad u_i=\frac{\exp(\mu_i)}{\sum_{l=1}^{d}\exp(\mu_l)}u.$$

Corollary 4.8 (Comonotonic Pareto model)**.** *For comonotonic risks of Pareto distributions of the same shape parameter α: $X_i\sim Pa(\alpha,\lambda_i)$, the optimal allocation by minimization of the two risk indicators is:*

$$\forall i\in\{1,\ldots,d\},\quad u_i=\frac{\lambda_i}{\sum_{l=1}^{d}\lambda_l}u.$$

4.3. **The dependence impact with some copulas models.**

In this section, we study the impact of the dependence on the optimal capital allocation using some copulas (see Nelsen [19] for review on copulas). The idea is to find the optimal allocation as function of the copula parameters in each case.

4.3.1. *FGM Bivariate Model.*

Let X_1 and X_2 be two risks of marginal exponential distributions $X_i\sim\exp(\beta_i)$ and FGM bivariate dependence structure with $-1\leq\theta\leq1$ as parameter (see Nelsen [19],Example 3.12., section 3.2.5). We assume that $\beta_1<\beta_2/2$.

In this case, the copula Pearson correlation coefficient is given by $\rho_P=\frac{\theta}{4}$, and the bivariate distribution function is:

$$F_{X_1,X_2}(x_1,x_2)=(1-e^{-\beta_1x_1})(1-e^{-\beta_2x_2})+\theta(1-e^{-\beta_1x_1})(1-e^{-\beta_2x_2})e^{-\beta_1x_1}e^{-\beta_2x_2}.$$

Proposition 4.9 (The optimal capital allocation for the indicator I in the FGM Model)**.** *For the indicator I, the optimal allocation of a capital u is given by $(\beta u, (1-\beta)u)$ such that $\beta = u_1/u$ is the unique solution in $[0,1]$ of the following equation:*

$$\begin{aligned}
&(1+2\theta)(h(\beta)-h(\alpha-\alpha\beta))+2\theta(h(2\beta)-h(2\alpha-2\alpha\beta)) \\
&+(1+\theta)h(\alpha+\beta-\alpha\beta)+\theta h(2\alpha+2\beta-2\alpha\beta)-\theta h(\alpha+2\beta-\alpha\beta)-\theta h(2\alpha+\beta-2\alpha\beta) \\
&=\frac{1+\theta}{\alpha-1}(h(\alpha)+\alpha h(1))+\frac{\theta}{\alpha-1}(h(2\alpha)+\alpha h(2))-\frac{\theta}{\alpha-2}(2h(\alpha)+\alpha h(2))-\frac{\theta}{2\alpha-1}(h(2\alpha)+2\alpha h(1))
\end{aligned}$$

where, h is the function $h(x)=\exp(-\beta_1 ux)$,and $\alpha=\beta_2/\beta_1$.

Remark 4.10. In the case of $\theta = 0$, we find exactly Equation (3.1) given by Proposition 3.1 for the independent exponential distributions model.

Equation (4.9) gives the behavior of the optimal allocation with respect to θ. It may be solved numerically.
Figure 1 presents an illustration of the optimal allocation variation with respect to the dependence parameter of the FGM copula.

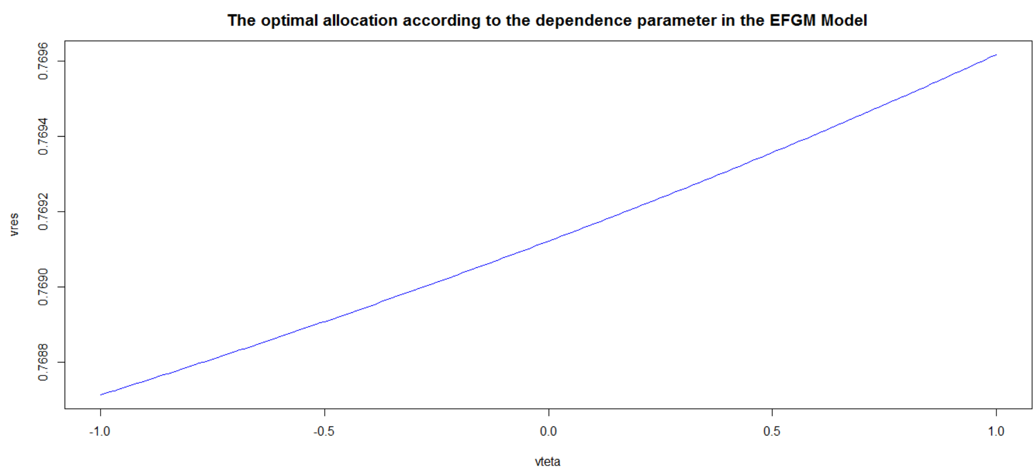

FIGURE 1. β as a function of θ. Case : $\beta_1 = 0.05$, $\beta_2 = 0.25$, and $u = 50$

We remark that β is an increasing function of θ, this can be verified analytically using the implicit function theorem.

4.3.2. *Marshall-Olkin Model.*
Let $Y_i \sim \exp(\lambda_i)$, with $i = 0, 1, 2$ be three independent random variables.
We construct two random variables with common shock: $X_i = \min(Y_i, Y_0)$ for $i = 1, 2$. X_i's have exponential marginal distributions of parameters $\beta_i = \lambda_i + \lambda_0$ (see e.g. Nelsen [19] section 3.1.1.). This dependence construction model have as Pearson correlation coefficient $\rho_P = \frac{\lambda_0}{\lambda_0+\lambda_1+\lambda_2}$.
The joint distribution function is given by:

$$\begin{aligned}
\bar{F}_{X_1,X2}(x_1,x_2) &= \mathbb{P}(X_1 > x_1, X_2 > x_2) = \mathbb{P}(Y_1 > x_1, Y_2 > x_2, Y_0 > \max(x_1,x_2)) \\
&= e^{-\lambda_1 x_1}e^{-\lambda_2 x_2}e^{-\lambda_0 \max(x_1,x_2)} \\
&= e^{-(\lambda_0+\lambda_1)x_1}e^{-(\lambda_0+\lambda_2)x_2}e^{\lambda_0 \min(x_1,x_2)} \\
&= \bar{F}_{X_1}(x_1)\bar{F}_{X_2}(x_2)e^{\lambda_0 \min(x_1,x_2)},
\end{aligned}$$

and the joint density function is the following:

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} f^1_{X_1,X_2}(x_1,x_2) = \beta_1 e^{-\beta_1 x_1}(\beta_2-\lambda_0)e^{-(\beta_2-\lambda_0)x_2} & si \quad x_1 > x_2 \\ f^2_{X_1,X_2}(x_1,x_2) = (\beta_1-\lambda_0)e^{-(\beta_1-\lambda_0)x_1}\beta_2 e^{-\beta_2 x_2} & si \quad x_1 < x_2 \\ f^0_{X_1,X_2}(x_1,x_2) = \lambda_0 e^{-\beta_1 x}e^{-\beta_2 x}e^{\lambda_0 x} & si \quad x_1 = x_2 = x \end{cases}.$$

Proposition 4.11 (The optimal capital allocation for the indicator I in the Marshall-Olkin Model)**.** *We suppose that $\lambda_1 < \lambda_2$. The optimal allocation of a capital u minimizing the indicator I is given by $(\beta u, (1-\beta)u)$, such that $\beta = u_1/u$ is the unique solution in $[0,1]$ of the following equation:*

$$g(\beta_2(1-\beta)) - g(\beta_1\beta) + \frac{\beta_1}{\beta_1-\lambda_2}g((\beta_1-\lambda_2)\beta+\lambda_2) + \frac{\lambda_2}{\beta_1-\lambda_2}g((\lambda_2-\beta_1)(1-\beta)+\beta_1)$$
$$-\frac{\lambda_1}{\lambda_1-\beta_2}g(\beta_2) = \frac{\lambda_2}{\beta_1-\lambda_2}g(\beta_1) + g(\lambda_s/2)[\frac{\beta_1}{\beta_1-\lambda_2} - \frac{\lambda_1}{\lambda_1-\beta_2}],$$

where, $\lambda_s = \lambda_0+\lambda_1+\lambda_2$, and g is the function $g(x) = \exp(-ux)$.

Remark 4.12. In the case of : $\lambda_0 = 0$ which is the independence case, we find exactly Equation (3.1) given by Proposition 3.1.

We can consider λ_0 as a dependence parameter in this model. Figure 2 presents an illustration of the variation of β as a function of λ_0.

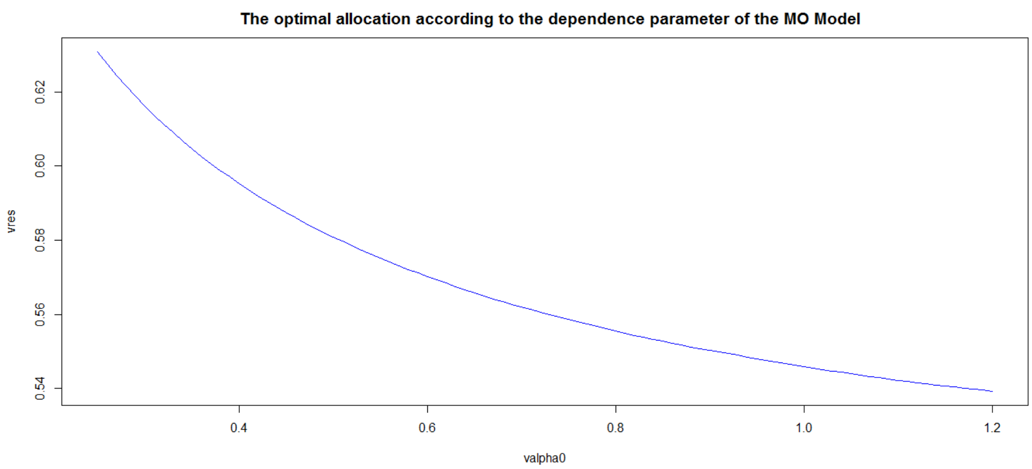

FIGURE 2. β as a function of λ_0 : $\beta_1 = 0.05$, $\beta_2 = 0.25$ and $u = 50$.

One can notice that β is a decreasing function of λ_0, that is coherent with the increase of β as function of $\alpha = \frac{\beta_2}{\beta_1} = \frac{\lambda_2+\lambda_0}{\lambda_1+\lambda_0}$ demonstrated in [9] in the independence case, since the two risks are independent, conditionally to Y_0.

Conclusion

Compared to conventional capital allocation methods, the main advantage of the capital allocation by minimizing multivariate risk indicators, is that we do not need to choose a univariate risk measure. It seems more coherent in a multivariate framework to use directly a multivariate risk indicator, not only for risk measurement, but also for capital allocation.

In this article, we have shown that this capital allocation method can be considered as coherent from an economic point of view. We also proved, for some specific models, that it fully takes into account the dependence between different risks. This method also illustrates the importance of the risky business portfolio choice and its impact on the management of the overall company capital.

We have studied the allocation asymptotic behaviour based on the level of the group capital. It has enabled us to build an idea of the capital level impact on the sensitivity of its allocation between branches. The comparison between the asymptotic optimal allocation in the case of sub-exponential and exponential distributions, serves to underscore the impact of the risks nature on the behavior of the allocation for a very large capital.

This method can be developed if one can construct some broader sets of multivariate risk indicators as this is the case for univariate risk measures.

Finally, the choice of a capital allocation method remains a complex and crucial exercise because some methods may be better suited to deal with specific issues, others can lead to dangerously wrong financial decisions. In the case of the proposed optimal capital allocation, the risk management is at the heart of the allocation process, and the company can allocate its capital and reduces its overall risk at the same time. Its risk aversion is reflected by the choice of the multivariate risk indicator to minimize. That is why we think that from risk management point of view, this method can be considered as more flexible.

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Appendix A. Some Lemmas and Proofs

A.1. **Proposition 4.9.**

Proof. The bivariate density function is the following:

$$\begin{aligned} f_{X_1,X_2}(x_1,x_2) &= (1+\theta)f(x_1,x_2,\beta_1,\beta_2)+\theta f(x_1,x_2,2\beta_1,2\beta_2) \\ &\quad -\theta f(x_1,x_2,2\beta_1,\beta_2)-\theta f(x_1,x_2,\beta_1,2\beta_2) f_{X_1,X_2}(x_1,x_2), \end{aligned}$$

where f is the function $f(x,t,a,b)=abe^{-(a-b)x}e^{-bt}$.
We use the equality: $f_{X_1,S=X_i+X_2}(x_1,s)=f_{X_1,X_2}(x_1,s-x_1)\mathbb{1}_{\{s\geq x_1\}}$ for all $s\geq x_1$, to find the expression of $F_{X_1,S}(x_1,s)$, using a double integration:

$$\begin{aligned} F_{X_1,S}(x_1,s) &= \int_0^s\int_0^{x_1} f_{X_1,X_2}(x,t-x)\mathbb{1}_{\{t\geq x\}}dxdt = \int_0^{x_1}\int_x^s f_{X_1,X_2}(x,t-x)dtdx \\ &= (1+\theta)F(x_1,s,\beta_1,\beta_2)+\theta F(x_1,s,2\beta_1,2\beta_2)-\theta F(x_1,s,2\beta_1,\beta_2)-\theta F(x_1,s,\beta_1,2\beta_2), \end{aligned}$$

where F is the following function:

$$F(x_1,s,a,b)=\int_0^{x_1}\int_x^s f(x,t,a,b)dtdx = 1-e^{-ax_1}+\frac{a}{b-a}e^{-bs}-\frac{a}{b-a}e^{-bs+(b-a)x_1}.$$

The same way and by the symmetry of the FGM model:

$$F_{X_2,S}(x_2,s)=(1+\theta)F(x_2,s,\beta_2,\beta_1)+\theta F(x_2,s,2\beta_2,2\beta_1)-\theta F(x_2,s,2\beta_2,\beta_1)-\theta F(x_2,s,\beta_2,2\beta_1).$$

Using $\mathbb{P}(X_i>u_i,S\leq u)=\mathbb{P}(S\leq u)-\mathbb{P}(X_i\leq u_i,S\leq u)$, the optimal allocation is the unique solution in $\mathcal{U}_u^2$ of the equation: $F_{X_1,S}(u_1,u)=F_{X_2,S}(u_2,u)$. Then, the optimal allocation is determined by β the solution of the equation: $F_{X_1,S}(\beta u,u)=F_{X_2,S}((1-\beta)u,u)$.
Since,

$$\begin{aligned} F_{X_1,S}(\beta u,u) &= 1+4\theta-(1+2\theta)h(\beta)-2\theta h(2\beta) \\ &\quad +(1+\theta)\frac{1}{\alpha-1}[h(\alpha)-h(\alpha+\beta-\alpha\beta)]+\theta\frac{1}{\alpha-1}[h(2\alpha)-h(2\alpha+2\beta-2\alpha\beta)] \\ &\quad -\theta\frac{2}{\alpha-2}[h(\alpha)-h(\alpha+2\beta-\alpha\beta)]-\theta\frac{1}{2\alpha-1}[h(2\alpha)-h(2\alpha+\beta-2\alpha\beta)], \end{aligned}$$

and,

$$\begin{aligned}F_{X_2,S}((1-\beta)u,u) &= 1+4\theta-(1+2\theta)h(\alpha(1-\beta))-2\theta h(2\alpha(1-\beta))\\&+(1+\theta)\frac{\alpha}{1-\alpha}[h(1)-h(\alpha+\beta-\alpha\beta)]+\theta\frac{\alpha}{1-\alpha}[h(2)-h(2\alpha+2\beta-2\alpha\beta)]\\&-\theta\frac{2\alpha}{1-2\alpha}[h(1)-h(2\alpha+\beta-2\alpha\beta)]-\theta\frac{\alpha}{2-\alpha}[h(2)-h(\alpha+2\beta-\alpha\beta)],\end{aligned}$$

we deduce from that the equation presented in the proposition 4.9. □

A.2. **Proposition 4.11.**

Proof. The joint distribution function is given by:

$$\begin{aligned}F_{X_1,S}(x_1,s) &= \int_0^s\int_0^{x_1} f_{X_1,S}(x,t-x)\mathbb{1}_{\{t>x\}}dxdt\\&=\int_0^s\int_0^{x_1} f^1_{X_1,S}(x,t-x)\mathbb{1}_{\{2x>t>x\}}dxdt+\int_0^s\int_0^{x_1} f^2_{X_1,S}(x,t-x)\mathbb{1}_{\{t>2x\}}dxdt\\&+\int_0^s\int_0^{x_1} f^0_{X_1,S}(x,t-x)\mathbb{1}_{\{t=2x\}}dxdt.\end{aligned}$$

we distinguish between two cases:
Case $s>2x_1$: in this case,

$$\begin{aligned}\int_0^s\int_0^{x_1} f^1_{X_1,S}(x,t-x)\mathbb{1}_{\{2x>t>x\}}dxdt &= \int_{2x_1}^s\int_0^{x_1} f^1_{X_1,S}(x,t-x)\mathbb{1}_{\{2x>t>x\}}dxdt\\&+\int_0^{2x_1}\int_0^{x_1} f^1_{X_1,S}(x,t-x)\mathbb{1}_{\{2x>t>x\}}dxdt\\&=\int_0^{2x_1}\int_0^{x_1} f^1_{X_1,S}(x,t-x)\mathbb{1}_{\{2x>t>x\}}dxdt\\&=\int_0^{2x_1}\int_{t/2}^{min(x_1,t)} f^1_{X_1,S}(x,t-x)dxdt\\&=\int_0^{x_1}\int_{t/2}^{t} f^1_{X_1,S}(x,t-x)dxdt+\int_{x_1}^{2x_1}\int_{t/2}^{x_1} f^1_{X_1,S}(x,t-x)dxdt,\end{aligned}$$

and,

$$\int_0^s\int_0^{x_1} f^2_{X_1,S}(x,t-x)\mathbb{1}_{\{t>2x\}}dxdt=\int_0^{x_1}\int_0^{s} f^2_{X_1,S}(x,t-x)\mathbb{1}_{\{t>2x\}}dtdx=\int_0^{x_1}\int_{2x}^{s} f^2_{X_1,S}(x,t-x)dtdx,$$

and,

$$\int_0^s\int_0^{x_1} f^0_{X_1,S}(x,t-x)\mathbb{1}_{\{t=2x\}}dxdt=\frac{\lambda_0}{\lambda_0+\lambda_1+\lambda_2}(1-e^{-(\lambda_0+\lambda_1+\lambda_2)x_1}),$$

then, we deduce the explicit expression of $F_{X_1,S}(x_1,s)$:

$$\begin{aligned}F_{X_1,S}(x_1,s) &= \frac{2\beta_1\lambda_2}{(\beta_1-\lambda_2)(\beta_1+\lambda_2)}(1-e^{-(\beta_1+\lambda_2)x_1})-\frac{\lambda_2}{\beta_1-\lambda_2}(1-e^{-\beta_1x_1})-\frac{\beta_1}{\beta_1-\lambda_2}(e^{-\beta_1x_1}-e^{-(\beta_1+\lambda_2)x_1})\\&+\frac{\lambda_1}{\lambda_1+\beta_2}(1-e^{-(\lambda_1+\beta_2)x_1})-\frac{\lambda_1}{\lambda_1-\beta_2}e^{-\beta_2s}+\frac{\lambda_1}{\lambda_1-\beta_2}e^{-(\lambda_1-\beta_2)x_1-\beta_2s}\\&+\frac{\lambda_0}{\lambda_0+\lambda_1+\lambda_2}(1-e^{-(\lambda_0+\lambda_1+\lambda_2)x_1}).\end{aligned}$$

Case $2x_1 > s > x_1$:

$$\int_0^s \int_0^{x_1} f^1_{X_1,S}(x,t-x)1\!\!1_{\{2x>t>x\}}dxdt = \int_0^s \int_{t/2}^{min(x_1,t)} f^1_{X_1,S}(x,t-x)dxdt$$
$$= \int_0^{x_1} \int_{t/2}^{t} f^1_{X_1,S}(x,t-x)dxdt + \int_{x_1}^s \int_{t/2}^{x_1} f^1_{X_1,S}(x,t-x)dxdt,$$

and,

$$\int_0^s \int_0^{x_1} f^2_{X_1,S}(x,t-x)1\!\!1_{\{t>2x\}}dxdt = \int_0^s \int_0^{t/2} f^2_{X_1,S}(x,t-x)dxdt,$$

and,

$$\int_0^s \int_0^{x_1} f^0_{X_1,S}(x,t-x)1\!\!1_{\{t=2x\}}dxdt = \frac{\lambda_0}{\lambda_0+\lambda_1+\lambda_2}(1-e^{-(\lambda_0+\lambda_1+\lambda_2)s/2}),$$

then, we deduce also in this case, the explicit expression of $F_{X_1,S}(x_1,s)$:

$$\begin{aligned}F_{X_1,S}(x_1,s) = &\frac{2\beta_1\lambda_2}{(\beta_1-\lambda_2)(\beta_1+\lambda_2)}(1-e^{-(\beta_1+\lambda_2)s/2}) - \frac{\lambda_2}{\beta_1-\lambda_2}(1-e^{-\beta_1 x_1})\\ &- \frac{\beta_1}{\beta_1-\lambda_2}(e^{-\beta_1 x_1} - e^{-(\beta_1-\lambda_2)x_1-\lambda_2 s}) + \frac{\lambda_1}{\lambda_1-\beta_2}(1-e^{-\beta_2 s})\\ &- \frac{2\lambda_1\beta_2}{(\lambda_1-\beta_2)(\lambda_1+\beta_2)}(1-e^{-(\lambda_1+\beta_2)s/2}) + \frac{\lambda_0}{\lambda_0+\lambda_1+\lambda_2}(1-e^{-(\lambda_0+\lambda_1+\lambda_2)s/2}).\end{aligned}$$

We remark that $\lambda_0+\lambda_1+\lambda_2 = \lambda_1+\beta_2 = \lambda_2+\beta_1$, and we suppose that $\lambda_1 > \lambda_2$. Using the monotony property, we deduce that $1 > \beta > 1/2$, then $2\beta u > u > \beta u$. So, for $u_1 = \beta u$, and $g(x) = \exp(-xu)$:

$$F_{X_1,S}(\beta u,u) = 1-g(\beta_1\beta) - \frac{\lambda_1}{\lambda_1-\beta_2}g(\beta_2) + \frac{\beta_1}{\beta_1-\lambda_2}g((\beta_1-\lambda_2)\beta+\lambda_2) + g(\lambda_s/2)[\frac{\lambda_1}{\lambda_1-\beta_2} - \frac{\beta_1}{\beta_1-\lambda_2}],$$

and,

$$F_{X_2,S}((1-\beta)u,u) = 1-g(\beta_2(1-\beta) + \frac{\lambda_2}{\lambda_2-\beta_1}[g((\lambda_2-\beta_1)(1-\beta)+\beta_1) - g(\beta_1)].$$

That is sufficient to get 4.11. □

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